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THE RANK OF ELLIPTIC CURVES

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§1. Introduction

Classically, methods of descent to obtain an upper bound for the rank of the Mordell-Weil group of an elliptic curve have depended on the existence of isogenies of degree two or three and explicit equations, or on the construction of coverings in ways suitable for computer calculation. (See [1], pp. 265–274 for a general survey.) In [5], Mazur introduced an elegant and effective method of descent using only mild information about the Néron minimal model, but requiring the existence of a rational isogeny.

This paper grew out of an effort to compute the rank of some elliptic curve factors of the Jacobians of modular curves left undone by Mazur in [6, 7] because they do not admit rational isogenies. It revolves around the general question of what can be said about an elliptic curve, given only its conductor and perhaps some supplementary information about the Néron model.

After some preliminaries in the next section, this paper splits into three parts. The first part, in section 3 and 4, contains the local descent. The remaining parts treat global questions.

The second part, in sections 5 and 6, may be read independently and performs a dual task. An analysis of two and three-division fields of a hypothetical curve, based on [16], gives information about the Néron model for use in the descent. This simultaneously provides criteria for the non-existence of elliptic curves having certain square-free conductors, extending nonexistence results due to Ogg [14], Coates [2], Setzer [17], Neumann [12], and others. The novelty here is that Diophantine considerations arising in the usual treatments are thrown back in a natural way to the arithmetic of certain number fields. As a result, we succeed in showing that for all primes $N < 1000$ where there is no known curve of conductor N, in fact no such curve exists, except for 8 values of N where the situation is still undecided. Since the writing of this paper R. Bölling has eliminated $N = 211$ and $N = 397$ by ad hoc methods.

The third part, in section 7, gives a rather precise upper bound for the Selmer group and hence for the rank of elliptic curves. We illustrate our results by obtaining bounds for the rank of all known and unknown curves of prime conductor $N < 1000$. (See Table III). These bounds are at most 1, except for 8 conductors where the bound is 2. Our bounds suggest that, in general, curves of prime conductor have the smallest rank compatible with the parity predictions of Birch and Swinnerton-Dyer.

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