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SCHUR MULTIPLIERS

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1. Introduction. If $A = (a_{jk})$ and $B = (b_{jk})$ are matrices of the same size (finite or infinite), their *Schur product* is defined to be the matrix of element-wise products

$$A * B = (a_{jk}b_{jk}).$$

This concept was first investigated by Schur in his remarkable paper [33], and has since arisen in several different areas of Analysis: [28], [35], [36] (complex function theory); [20], [4] (Banach spaces); [37], [14] (operator theory); and [39] (multivariate analysis). The term "Hadamard product," coined by von Neumann and introduced in the literature by Halmos [13], has been adopted by many authors. There is, however, much justification for the term "Schur product," and we refer the reader to [39] for an historical discussion.

The purpose of this paper is to study the behavior, under Schur multiplication, of the norms $\|\cdot\|_{p,q}$, $1 \le p, q \le \infty$, where

$$(1) \qquad \|A\|_{p,q} = \sup_{\|x\|_p \le 1} \left(\sum_j \left|\sum_k a_{jk}x_k\right|^q\right)^{1/q}$$

denotes the usual operator norm of $A : l^p \to l^q$. Our motivation stems, of course, from a desire to understand better the norms (1). In particular, we shall be interested in *(p, q)-multipliers*: matrices M for which $M * A$ maps l^p into l^q whenever A does. Several of the above results may be phrased in these terms, and our general approach serves to clarify, and sometimes to improve upon, the results of [4], [14], [20], [33]. (See, for example, 2.2, 6.4, 6.5, 7.1 and 8.1.) There are close connections between Schur multipliers and the theory of absolutely summing operators (Theorem 4.3 and the remarks preceding Theorem 7.1), and we take advantage of this fact whenever convenient. On the other hand, many of our results (in particular 5.2, 7.1, 7.4, 8.1 and 9.3) may be viewed as giving new information on absolutely summing operators.

We begin, in Section 2, by showing that the matrix transformations from l^p to l^q form a commutative Banach algebra under Schur multiplication (Theorem 2.2). These algebras turn out to be quite interesting and have been studied in some detail by Q. Stout [38]. The special case, $p = q = 2$, was obtained by Schur ([33], Theorem III), but the proof given here is quite different, our auxiliary inequalities (Proposition 2.1) requiring weaker hypotheses than his.

In Section 3 we show that the estimates of Proposition 2.1 are the best pos-

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