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THE TEICHMÜLLER DISTANCE IS DIFFERENTIABLE

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Introduction. Let $T(p, n)$ be the Teichmüller space of marked Riemann surfaces of finite tpye (p, n). The purpose of this paper is to prove that the Teichmüller distance is a continuously differentiable function on the complement of the diagonal in $T(p, n) \times T(p, n)$ and to give a formula for its differential. We prove our formula in §2. The continuity of the differential, which is a key ingredient of our proof, is a consequence of a theorem about the cotangent bundle of $T(p, n)$, which we prove in §1. In §3 we apply our formula to the study of an extremal problem for quasiconformal mappings, first formulated and solved by Bers [8]. Our Theorem 3.3 is a slight improvement of one of Bers' results.

The question of the differentiability of the Teichmüller distance came to our attention several years ago, during an unsuccessful attempt to understand Saul Kravetz's paper [13] on the geometry of Teichmüller spaces. Aside from a small misprint (G should be replaced by $\bar{G}$); formula (7) on p. 22 of [13] gives the differential of the Teichmüller distance on $T(p, 0)$. Unfortunately that formula occurs in the middle of an attempt to prove the false theorem that the curvature of $T(p, 0)$ is negative, and its proof is incorrect. Linch was able to obtain partial results in her thesis [14] by lengthy computations, again only for the spaces $T(p, 0)$.

Since direct computation is discouragingly hard, we have relied instead on the method of wishful thinking. We assume in §2.2 that the distance function is differentiable at a given point x, and we investigate its differential. It turns out that we can find the differential explicitly and that it is the value at x of a globally defined continuous differential one form. Since the distance function is locally Lipschitz, general regularity properties of Lipschitz functions imply that the distance function is continuously differentiable.

It is well known (see Royden [16] and O'Byrne [15]) that the Teichmüller distance is induced from a Finsler structure on $T(p, n)$. Royden [16] proved that the Finsler norm defines a continuously differentiable function on the space of non-zero tangent vectors. That fact does not seem to simplify the proof of our Theorem 2.1. The author does not know any general relationship in Finsler spaces between smoothness of the nor m function and smoothness of the induced distance function.

We have assumed that the reader has some familiarity with Teichmüller

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