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ON THE DERIVATIVES OF THETA FUNCTIONS AND MODULAR FORMS

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Dedicated to Kenkichi Iwasawa on the occasion of his sixtieth birthday

The algebraicity of an analytically defined object is a fascinating subject both in number theory and in algebraic geometry, but has attracted unaccountably few researchers. For instance, it seems that there are more mathematicians who deal with the transcendency of the special values of analytic functions than those who prove the algebraicity. In the present work, we shall discuss several problems of algebraicity which involve differentiation. We divide the paper into four sections, the first of which is an investigation of differential forms and derivations on an abelian variety A parametrized by classical theta functions. We first observe that if a point w on the Siegel upper half space and a matrix δ of elementary divisors which determines the type of polarization are given, then the variety A is defined over the field $\mathfrak{K}_S[w]$ generated by the values $f(w)$ for all f in a certain field $\mathfrak{K}_S$ of Siegel modular functions. Here $\mathfrak{K}_S$ consists of the quotients of modular forms with rational Fourier coefficients with respect to a certain congruence subgroup Γ_δ of $\mathrm{Sp}(n, \mathbf{Z})$, depending on δ. Next, we shall give an explicit basis of holomorphic 1-forms on A rational over $\mathfrak{K}_S[w]$ by means of the derivatives of theta functions. This will lead to an answer to the question concerning the periods of integrals rational over a field of definition, which Weil posed in his recent article [8].

The second section, for which the first one may be considered preliminaries, is a continuation of the previous paper [7], where we showed that a classical theta function with complex multiplication, modified by a certain exponential factor and a constant factor, takes algebraic values on the points commensurable with the periods. This result will be extended to certain non-holomorphic derivatives of such functions. The behavior of the special values under automorphisms will also be determined. These are closely connected with the results of another previous paper [5], in which certain non-holomorphic derivatives of Hilbert modular forms are discussed. The connection will be clarified in the third section, by viewing A as a member of the family of abelian varieties with a fixed totally real algebraic number field as their endomorphism algebras. We shall first prove the analogues of the theorems of §1 for such A. Then it will be shown that the heat equation satisfied by theta functions implies the coincidence of the two types of derivatives at the origin of A. An observation of the same kind will be made for Siegel modular forms in the last section, in

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