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TOEPLITZ OPERATORS WITH SEMI-ALMOST PERIODIC SYMBOLS

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The present note concerns the Fredholm theory of Toeplitz operators on H^2 of the upper half-plane whose symbols lie in the algebra generated by AP (the space of Bohr almost periodic functions on $\mathbf{R}$) and PC_∞ (the space of continuous functions on $\mathbf{R}$ that have finite limits at $+\infty$ and at $-\infty$). We denote the above algebra by SAP. The analysis below carries further the analyses by Coburn and Douglas [4], Douglas [5], [7], and Gohberg and Feldman [10] of Toeplitz operators whose symbols lie in AP and in the algebra generated by AP and $C = \{u \in PC_\infty : u(+\infty) = u(-\infty)\}$.

Some definitions are needed before results can be stated. We denote the L^∞ space of Lebesgue measure on $\mathbf{R}$ by L^∞. For f in L^∞, the Toeplitz operator on H^2 induced by f will be denoted by T_f.

For f and g in L^∞ we let

$$\operatorname*{dist}_{+\infty}(f, g) = \operatorname{ess}\limsup_{t\to+\infty} |f(t) - g(t)|,$$

$$\operatorname*{dist}_{-\infty}(f, g) = \operatorname{ess}\limsup_{t\to-\infty} |f(t) - g(t)|,$$

$$\operatorname*{dist}_{\infty}(f, g) = \operatorname{ess}\limsup_{|t|\to\infty} |f(t) - g(t)|.$$

If f and g are in AP, the above three quantities all equal $\|f - g\|_\infty$.

We fix a function u_+ in PC_∞ which has values in $[0, 1]$ and satisfies $u_+(+\infty) = 1$ and $u_+(-\infty) = 0$, and we let $u_- = 1 - u_+$. It is easily seen that each function f in SAP can be written uniquely as $f = u_+f_+ + u_-f_- + f_0$, where f_+ and f_- are in AP and f_0 is in C_0 (the space of functions in C that vanish at ∞). We have $\operatorname{dist}_{+\infty}(f, f_+) = 0 = \operatorname{dist}_{-\infty}(f, f_-)$. If $(f_n)_{n=1}^\infty$ is a uniformly convergent sequence in SAP with limit f, then we see from the above that the sequences $((f_n)_+)_{n=1}^\infty$ and $((f_n)_-)_{n=1}^\infty$ are uniformly convergent, say to f_+ and f_-. The latter functions belong to AP, and $f - u_+f_+ - u_-f_-$ is clearly in C_0, so that f is in SAP. Thus, SAP is uniformly closed and so is a C^*-algebra. In particular, a function in SAP is invertible if and only if it is bounded away from 0.

The following easily verified observation is recorded for future reference.

Lemma 0. *The maps $f \to f_+$ and $f \to f_-$ are *-homomorphisms of* SAP *onto* AP.

For f in AP we let $M(f)$ denote the Bohr mean value of f [1]. Thus, for f

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