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NOTES ON EXTENSIONS OF C^*-ALGEBRAS

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In a recent paper [22], Dan Voiculescu has obtained several new results relating to compact perturbations of representations of C^*-algebras. In particular, he has shown that the semigroup Ext(A) of all (classes of) extensions of the compact operators by a given separable C^*-algebra A possesses a neutral element.

On the other hand, in [12] Choi and Effros have proved that every *-homomorphism of a separable nuclear C^*-algebra has a completely positive linear lifting. These two results lead easily to a proof that Ext(A) is a group whenever A is separable and nuclear. This, one of the main results of Brown, Douglas and Fillmore [8] has now been generalized so as to cover most cases of interest.

The original purpose of these notes was to give an exposition of Voiculescu's theorem which seemed to us more natural than the original, to relate Voiculescu's results to some work of Bunce and Salinas [10] on matrix ranges of operators, and to point out the key role of "quasicentral" approximate units for problems of this nature. Subsequently, it became clear that the same methods could be used to simplify the proof of the Choi–Effros lifting theorem. We have therefore expanded the original version so as to include a discussion of liftings, together with a brief discussion of Ext(A).

There are some new results below, and perhaps the methods have some novelty. But the main theorems were found by other mathematicians, and this paper is at least semi-expository.

Finally, we want to thank Joe Stampfli for making two useful suggestions.

1. Quasicentral approximate units.

Let $\mathcal{H}$ be a separable Hilbert space. Recall that an operator T on $\mathcal{H}$ is called quasidiagonal if there is a sequence F_n of finite dimensional projections such that $F_n \uparrow 1$ and $\|F_nT - TF_n\| \to 0$. It is not very hard to see that this is equivalent to the existence of a mutually orthogonal sequence E_n of finite dimensional projections such that $\sum E_n = 1$ and $T = \sum E_nTE_n + K$, where K is compact; i.e., T is a compact perturbation of a block diagonal operator (see [18]).

It is well-known that there exist operators which are not quasidiagonal. In this section we will prove a general result which implies, nevertheless, that one can always achieve the above conditions $F_n \uparrow 1$ and $\|F_nT - TF_n\| \to 0$ with finite rank positive operators F_n. We also show that there exist positive

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