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SMALL SALEM NUMBERS

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1. Introduction. There is a great deal of detailed knowledge concerning the set S of Pisot–Vijayaraghavan numbers. (For undefined terms, see section 2). For example S is known to be closed [13], all the points in S less than $\tau = (\sqrt{5} + 1)/2$ are known [5], and there is even considerable information about the derived sets of S [1], [6].

In contrast very little is known about the set T of Salem numbers. The principal result here is due to Salem [14] who showed that each point of S is a limit point, from both sides, of the set T.

Salem's proof of this fact used a construction which we describe in more detail in section 3. By this construction, given θ in S, two infinite sequences θ_n^+ and θ_n^- of points of T are produced which converge to θ. Our main result (Theorem 4.1) is that Salem's construction in fact produces all members of T. Some implications of this are discussed in sections 5 and 6. Unfortunately, this is not yet enough to show that the only limit points of T are the points of S, but Theorem 6.1 shows what else is required in order to decide this.

The proof of Theorem 4.1 shows that any given σ in T is produced infinitely often by arbitrarily large θ in S. Proposition 5.1 shows that, for $n \geq 2$, θ_n^+ and θ_n^- tend to infinity with θ. More precise information can be obtained using the methods of Dufresnoy and Pisot [5], but this will be presented elsewhere along with other applications of their methods.

Since the smallest number in S is $\theta_0 \simeq 1.3247$, as shown by Siegel [16], Salem's result shows that any interval $(0, a)$ with $a \geq \theta_0$ contains infinitely many Salem numbers. Thus there is a particular interest in "small" Salem numbers which we define here to be those less than 1.3. In section 7 we give a list of 39 such numbers, all those that we are currently aware of. These were constructed by a variant of Salem's method which uses more general polynomials as a starting point. (It is not particularly efficient to use his method directly since, in general, only θ_1^+ or θ_1^- can be a small Salem number and it often happens that $\theta_1^- = 1$.)

There is a relation between the search for small Salem numbers and a conjecture of Lehmer [10, p. 476]. He asks whether there is a universal constant $\epsilon_0 > 0$ such that the following holds. Let P be a monic polynomial with integer coefficients and let Ω be the absolute value of the product of those roots of P which lie outside the unit circle (assuming there is at least one such root);

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