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THE LEMMA OF THE LOGARITHMIC DERIVATIVE IN SEVERAL COMPLEX VARIABLES

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1. Introduction.

In 1925, R. Nevanlinna [9] proved what is known as the lemma of the logarithmic derivative for a meromorphic function f defined on the complex plane:

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \leq a_1 + a_2 \log (r) + a_3 \log T_f(r)$$

for all r outside a union of intervals of finite Lebesgue measure, where T_f is the characteristic or order function of f. This lemma was the main technical tool in the proof of his celebrated defect relations and it has served a similar function in much of the subsequent work in value distribution theory.

The main purpose of this paper is to prove the lemma of the logarithmic derivative for meromorphic functions of several variables. Our proof uses the method of negative curvature, a general technique which was implicit in a proof of the lemma given by F. Nevanlinna ([10], page 247) and which was explicitly developed by Carlson and Griffiths [2] and later in Griffiths–King [5] and Cowen–Griffiths [3]. More particularly, our proof is modeled on proposition 9.3 of [5]. It is given in section 4 along with some corollaries which state, roughly, that the order functions of the derived maps of a meromorphic map $f : \mathbf{C}^n \to \mathbf{P}^m$ are no larger than constant multiples of the order function of f.

In section 5, we use our theorem to give a simple proof of the defect relation for meromorphic maps $f : \mathbf{C}^n \to \mathbf{P}^m : \sum_{i=1}^{q} \delta_f(H_i) \leq m + 1$ for $H_1 , \cdots , H_q$ hyperplanes in $\mathbf{P}^m$ in general position. This proof follows exactly the one given by Henri Cartan for meromorphic curves in 1932 [2A]. The author wishes to thank B. Shiffman for his insights concerning this proof.

In section 2 we mention facts about meromorphic maps and currents and in section 3 we develop the requisite Nevanlinna theory, from the point of view of currents. In particular, we give, following [12], a complete proof of the equation of currents (3.3) which is basic to the method of negative curvature and to the proof of our main theorem. Many of the definitions and conclusions of section 3 have been specialized to our case of interest, i.e., meromorphic maps $f : \mathbf{C}^n \to \mathbf{P}^m$ and the hyperplane bundle over $\mathbf{P}^m$.

2. Preliminaries.

(a) *Meromorphic maps.* A *meromorphic map* $F : \mathbf{C}^n \to \mathbf{P}^m$ is a holomorphic

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