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CHARACTERIZATION OF PSEUDODIFFERENTIAL OPERATORS AND APPLICATIONS

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In this paper we give affirmative answers to the following interrelated questions:

Can one characterize the pseudodifferential operators of a given class among the maps from $\mathcal{S}$ to $\mathcal{S}'$?

If a zero order pseudodifferential operator in L^2 has a bounded inverse, is the inverse a pseudodifferential operator of the same class?

Do the Hilbert spaces obtained by interpolating between the weighted Sobolev spaces of [1], [2] belong to the same family of weighted Sobolev spaces?

If a partial differential or pseudodifferential operator satisfies a priori estimates guaranteeing hypoellipticity, as in [1], [2], does there necessarily exist a left parametrix? To illustrate, we show that certain partial differential operators of order $m \geq 2$ having all (real) characteristics double and satisfying an estimate with loss of one derivative have a parametrix of type $L_{\frac{1}{2},\frac{1}{2}}{}^{1-m}$. (Various cases of this are known.)

Operators of type $L_{\rho,\rho}{}^{0}$ are characterized in section 1, and operators of general type in section 2. The characterization is in terms of boundedness of certain commutators. The results on inverses and some applications are given in section 3. Sections 4 and 5 are independent of each other. In section 4 we show that a complex power of a positive selfadjoint psuedodifferential is an operator of the same type and appropriate order. The result on interpolation is a consequence. The link between a priori estimates and parametrices is given in section 5. Application is made to the double characteristic result mentioned above, and also to a construction of Sjöstrand in a case of higher characteristics.

1. Characterization of operators of type (ρ).

By a *symbol* we mean any smooth function on $R^n \times R^n = \{(x, \xi)\}$. If a is a symbol we denote mixed partial derivatives by

$$a_{(\beta)}{}^{(\alpha)} = \partial_x{}^{\beta} D_{\xi}{}^{\alpha} a.$$

The operator $A = a(x, D)$ corresponding to a is defined by

$$Au(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi)\, d'\xi \tag{1.1}$$

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