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EMBEDDING RIEMANN SURFACES IN POLYDISCS

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1. Introduction. Let $\bar{W} = W \cup \Gamma$ be a compact bordered Riemann surface with interior W and non-empty border Γ. Let U^n denote the open unit polydisc in $\mathbf{C}^n$, and let $z = (z_1, \cdots, z_n)$ denote the coordinates of $\mathbf{C}^n$. Let $H^\infty(W)$ and $H^\infty(U^n)$ be the Banach algebras of bounded analytic functions on W and on U^n. If $\Phi : W \to U^n$, $\Phi = (\varphi_1, \cdots, \varphi_n)$, is a holomorphic mapping, then there is an associated restriction operator $R_\Phi : H^\infty(U^n) \to H^\infty(W)$ defined by $R_\Phi(f) = f \circ \Phi$, for f in $H^\infty(U^n)$. We say that Φ admits an *extension operator* if there is a mapping $E_\Phi : H^\infty(W) \to H^\infty(U^n)$ such that $R_\Phi E_\Phi(f) = f$, for all f in $H^\infty(W)$. We shall prove that Φ admits a norm-preserving extension operator if and only if one component of Φ is a conformal mapping of W onto a disc. Our methods yield the corresponding theorem for holomorphic mappings from W into the ball, and also a similar theorem of Rudin [6] for holomorphic mappings of one polydisc into another.

Our proof uses the infinitesimal forms of the Caratheodory and Kobayashi metrics; in particular we need some information about the associated extremal functions and mappings. If Φ admits a norm-preserving extension operator, then it is an isometry with respect to the Caratheodory metric. Thus the solutions of certain extremal problems are preserved. Our theorem then follows from results of Ahlfors [1] on extremal problems for Riemann surfaces.

2. The Caratheodory and Kobayashi metrics in a polydisc. We shall compute the infinitesimal forms of the Caratheodory and Kobayashi metrics in a polydisc, paying special attention to the description of the extremal mappings and functions. The computation of the metrics, along with much related material, can be found in Kobayashi [2]. Royden [5] computes the infinitesimal form of the Kobayashi metric, and Reiffen [3] computes the infinitesimal form of the Caratheodory metric.

Royden's definition of the Kobayashi metric is as follows: Let M be a complex manifold, p a point of M and t a holomorphic tangent vector at p. Let U be the open unit disc in the complex plane, let ζ be the coordinate in U, and let x denote differentiation with respect to ζ at the origin. Consider all mappings $\Psi : U \to M$ such that $\Psi(0) = p$ and $\Psi_*(x) = at$ for some a in $\mathbf{C}$. The length of t is defined to be

$$K_M(p, t) = \inf |a|^{-1}$$

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