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THE FATOU THEOREM FOR OPEN RIEMANN SURFACES

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Let $f(z)$ be a bounded (resp., quasibounded) harmonic function on the open unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane $\mathbf{C}$. Then the following statements are usually referred to as the Fatou theorem, which can be traced back to Fatou's celebrated paper [5]:

(a) For almost every $\omega \in [0, 2\pi)$ the function f admits a radial limit $\lim_{r\to 1} f(re^{i\omega})$ $(= f(e^{i\omega})$, say). The function $\omega \rightarrow f(e^{i\omega})$ is measurable, bounded (resp., integrable) and $f(z)$ is represented by the Poisson integral formula with the boundary data $f(e^{i\omega})$.

(b) For almost every $\omega \in [0, 2\pi)$ for which $f(e^{i\omega})$ exists, $f(z)$ tends uniformly to $f(e^{i\omega})$ as $z \in U$ tends to the point $e^{i\omega}$ through any sector $S(e^{i\omega}; \theta; \rho)$ with $0 < \theta < \pi/2$ and $0 < \rho < 1$, where $S(e^{i\omega}; \theta; \rho)$ denotes the set of $z \in U$ satisfying $-\theta < \arg(1 - ze^{-i\omega}) < \theta$ and $1 - \rho < |z| < 1$.

The purpose of this paper is to extend the statement (b) as well as some related results concerning boundary behaviors of analytic or harmonic functions to a class of hyperbolic Riemann surfaces. Let R be a hyperbolic Riemann surface and $G(a, z)$ the Green function for R with pole at a point $a \in R$. For any number $\alpha > 0$ we shall denote by $R(\alpha, a)$ the region $\{z \in R : G(a, z) > \alpha\}$ and by $B(\alpha, a)$ the first Betti number of $R(\alpha, a)$. We need the condition:

$$(W) \qquad \int_0^\infty B(\alpha, a)\, d\alpha < \infty \quad \text{for some (and hence all) } a \in R.$$

We shall show below that the Fatou theorem is valid for any hyperbolic Riemann surface R satisfying the condition (W). This means that we can define sectors in R with vertices at almost all points in the Martin boundary of R and show the statement (b) for this setup. The statement (a) has been extended to hyperbolic surfaces satisfying (W) in our papers [6, 7] and we are going to show that the same argument can be used to prove an extension of the statement (b). We note in passing that Doob [4] and Parreau [9] discussed extensions of the statement (a) to hyperbolic Riemann surfaces. As Professor Z. Kuramochi pointed out to us, Parreau [9] had treated the case of regular hyperbolic surfaces satisfying (W) by considering limits along the Green lines as a substitute for radial limits. In the paper [9] there is no mention about the relation between the Green lines and the Martin boundary, so that our results in [6, 7] extend Parreau's work. In Doob's extension, on the other hand, arbitrary hyperbolic surfaces are considered but radial limits are replaced by limits along Brownian paths; so his extension seems to have a flavor a little bit different from ours.

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