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THE EXTENSION OF THE WEIL-PETERSSON METRIC TO THE BOUNDARY OF TEICHMULLER SPACE

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1. Introduction and statement of the theorems.

The Teichmüller space T_g of a closed Riemann surface of genus $g \geq 2$ is a complex manifold with an intrinsic invariant Riemannian structure. The corresponding metric, the Weil-Petersson metric, is Kählerian and has negative holomorphic sectional curvature [2]. Recently Wolpert [8] and Chu [4] independently proved that the metric is not complete by showing that certain curves in T_g which leave every compact set have finite length.

In this paper we investigate the behavior of the Riemannian metric near the boundary spaces consisting of Riemann surfaces with nodes. These spaces have been discussed in detail by Bers [3] and Earle and Marden [5] and we refer to their papers for details.

To fix ideas let S_0 be a surface of genus g and $\gamma_1, \cdots, \gamma_p$ disjoint, nonhomotopic, noncontractible simple closed curves on S_0. Let X_0 be the Riemann surface with nodes which results from pinching each curve to a point. The boundary space $\delta_{\gamma_1}, \cdots, {}_{\gamma_p}T_g$ is a (possibly product) Teichmüller space of complex dimension $p' = 3g - 3 - p$. We parametrize a neighborhood of X_0 in the boundary by complex coordinates $\tau = (\tau_1, \cdots, \tau_{p'})$, $\tau = 0$ corresponding to X_0. In addition the coordinates $t = (t_1, \cdots, t_p)$ in a neighborhood of $0 \in \mathbf{C}^p$ may be used to parametrize the "blowing up" of the nodes on X_τ to form the Riemann surface with nodes $X_{t,\tau}$. The pairs $(t, \tau) \in \mathbf{C}^{3g-3}$ parametrize deformations of X_0 with the following properties. The complement of the hyperplanes $t_i = 0$ is the quotient of a domain in T_g by a subgroup of the modular group generated by Dehn twists. On the other hand for $J \subset \{1, \cdots, p\}$ the intersection of the hyperplanes $t_i = 0$, $i \in J$, is the quotient by the group of twists of a domain in the boundary Teichmüller space determined by pinching precisely the curves γ_i, $i \in J$.

We carry out this construction in detail in §2 relying on the recent results of Earle and Marden [5].

The boundary Teichmüller spaces themselves carry a Riemannian structure and a Weil-Petersson metric which is invariant under Dehn twists. Let

$$s = (s_1, \cdots, s_{3g-3}) = (t, \tau).$$

We represent all the metrics on the quotients by $\sum_{i,j=1}^{3g-3} g_{i\bar{j}}(s)\, ds_i \overline{ds_j}$. On each hyperplane $t_k = 0$, $g_{i\bar{j}}(s)$ is defined only for $i, j \neq k$.

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