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THE SELBERG TRACE FORMULA AND THE RIEMANN ZETA FUNCTION

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This paper is an expanded version of my recent John J. Gergen Lectures at Duke University. In order to keep the size down, I have decided to retain an expository (and somewhat informal) style.

1. Some preliminary remarks. My main objective is to draw attention to the fascinating interplay which exists between three areas of mathematics:

analytic number theory

(1.1)

eigenvalue problems on a compact Riemann surface ↔ the Selberg trace formula.

For purposes of exposition, it will be assumed that the reader has *some* familiarity with: (a) number theory; (b) the differential equations of mathematical physics; (c) discontinuous groups and Riemann surfaces. The following books are good references for these topics: [8, 19, 26, 31, 39, 41, 55, 66].

Whenever possible, my basic point of view will be that of analytic number theory. Experience has shown that the inter-relations in (1.1) are clearest, and most exciting, when viewed in this light.

The topics mentioned in (1.1) have one very important thing in common. Namely, in terms of style, the proofs in each area are characterized by hard estimates and careful attention to detail. In other words, the key to success in these areas is hard work coupled with good ideas. This is especially true in analytic number theory and trace formulas, where one's techniques are just as important as the theorems.

Since this paper is expository, our main task will be to present the "big" picture. This obviously provides a convenient *excuse* for emphasizing only the key ideas.

Most of the emphasis will be placed on trace formulas. In sections 4 and 5, we shall use analytic number theory to arrive at a very natural motivation for the trace formula. This seems to be the first time such an approach has appeared in print. The underlying philosophy is very simple: start out with the simplest specializations and gradually generalize. Cf. [22, pp. 296–297].

Although there may be some possibility of understanding the Riemann Hypothesis using trace formulas, the over-all state of affairs is still very hazy.

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