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STRUCTURE AND INTERPOLATION THEOREMS FOR CERTAIN LIPSCHITZ SPACES AND ESTIMATES FOR THE $\bar{\partial}$ EQUATION

STEVEN G. KRANTZ

The principal results of this paper are of two types. We first prove some structure and interpolation theorems for the non-isotropic Lipschitz spaces which have seen considerable use recently in the theory of several complex variables (c.f. [1], [3], [7], [12]). We then give an application of one of these results to the regularity problem for the $\bar{\partial}$ equation.

In particular, we shall be interested in Lipschitz regularity for the $\bar{\partial}$ operator. This question has already been considered, and significant results obtained, by Alt [1] and Siu [10]. Our results will contain theirs, but we will use some ideas from [10] to obtain them.

We now recall some of the notions connected with the $\bar{\partial}$ operator, and some of the function spaces we will need. In general, notation will be as in [7]. Recall that if $\mathcal{E} \subseteq \mathbf{R}^n$ then $\Lambda_\alpha(\mathcal{E})$ denotes the classical Lipschitz space on $\mathcal{E}$ (c.f. Stein [10]). Also, $C^k(\mathcal{E})$ denotes the space of functions with bounded, continuous derivatives up to order k under the norm

$$||f||_{C^k(\mathcal{E})} = \sum_{|\alpha| \leq k} \left|\left|\left(\frac{\partial}{\partial x}\right)^\alpha f\right|\right|_{L^\infty(\mathcal{E})}$$

If $\mathcal{E} \subset\subset \mathbf{C}^n$ is open with C^2 boundary, and $z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n$ denote the usual complex coordinate functions, then Λ_α and C^k may be defined in an obvious way in terms of complex coordinates. Let $W \supseteq b\mathcal{E}$ be a tubular neighborhood of $b\mathcal{E}$ with a smooth retraction to $b\mathcal{E}$ which we denote by $z \to \tilde{z}$. The choice of W is not important, but once chosen it is forever fixed. For $w \in b\mathcal{E}$ let ν_w denote the unit outward normal to $b\mathcal{E}$ and let $\{\mathbf{C}\nu_w\}$ be the one dimensional complex subspace of $\mathbf{C}^n$ generated by ν_w. If T_z denotes the complexified tangent space to the manifold $\mathcal{E} \subseteq \mathbf{C}^n$ at $z \in \mathcal{E} \cap W$, then we may regard $\{\mathbf{C}\nu_{\tilde{z}}\}$ as a subspace of T_z and we denote it by T_z^2. We let T_z^1 be the complement of T_z^2 in T_z with respect to the canonical Hermitian inner product. If $u \in C^1(\mathcal{E})$ we let $\text{grad}^t\, u(z)$ denote any normalized, real $(2n - 2)$-tuple of basis vectors for T_z^1 applied to u at z. Let $\mathcal{C}^k$ denote the class of C^k curves $\gamma : [0, 1] \to W \cap \mathcal{E}$ satisfying $|\gamma^{(j)}(t)| \leq 1$ for all $1 \leq j \leq k$ and $t \in [0, 1]$. Let $\mathcal{C}_1^k \subseteq \mathcal{C}^k$ be those curves satisfying $\dot{\gamma}(t) \in T^1_{\gamma(t)}$ for all $t \in [0, 1]$.

Definition 0.1. If $0 < \alpha \leq \beta < \infty$, $\mathcal{E}$ is as above, we define

$$\Gamma_{\alpha,\beta}(\mathcal{E}) = \{f : ||f||_{\Lambda_\alpha} + \sup_{\gamma \in \mathcal{C}_1^{[\beta]+1}} ||f \circ \gamma||_{\Lambda_\beta([0,1])} = ||f||_{\Gamma_{\alpha,\beta}} < \infty\}.$$

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