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EXAMPLES OF MIXING SEQUENCES

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1. Introduction. Let $\{X_n, n = 0, \pm 1, \pm 2, \cdots\}$ or $\{X_n, n = 0, 1, 2, \cdots\}$ be a stochastic sequence on a probability space $(\Omega, \mathfrak{B}, P)$. Let $\mathfrak{B}_i^j$ denote the σ-algebra of events generated by $\{X_i, X_{i+1}, \cdots, X_j\}$ (i may be $-\infty$, j may be ∞). Define the following quantities for $k = 1, 2, \cdots$.

(1) $$\alpha(k) = \sup \{|P(BC) - P(B)P(C)|\},$$

(2) $$\lambda(k) = \sup \{|P(C \mid B) - P(C)|\},$$

and

(3) $$\rho(k) = \sup \left\{\left|\frac{P(BC)}{P(B)P(C)} - 1\right|\right\},$$

where in each case the supremum is taken over all $m \in \mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$ (respectively $\mathbf{Z}^+ = \{0, 1, \cdots\}$ and all $B \in \mathfrak{B}_{-\infty}^m(\mathfrak{B}_0^m)$ and $C \in \mathfrak{B}_{m+k}^\infty$ with $P(B) > 0$ and $P(C) > 0$. We say $\{X_n\}$ is *strongly mixing* (*φ-mixing*, *ψ-mixing*) if $\alpha(k)$ (or $\lambda(k)$ or $\rho(k)$, respectively) $\to 0$ as $k \to \infty$. Since $\alpha(k) \leq \lambda(k) \leq \rho(k)$ for all k, ψ-mixing implies φ-mixing, which in turn implies strong mixing. A *mixing rate* is any condition which specifies the rate at which α, λ or ρ converges to zero.

In recent years, various limit theorems have been extended from sequences of independent random variables to random sequences satisfying one of the above mixing conditions with some specified mixing rate (see references [1], [2], [5]–[9]). These theorems usually assume mixing rates such as $\sum_{n=1}^\infty \rho(n)^\theta < \infty$ or $\sum_{n=1}^\infty n^2\lambda(n)^\theta < \infty$ for some particular $\theta > 0$.

The purpose of this note is to determine whether there exist random sequences such that a given mixing rate holds or fails. In Section 2, we show that any function α(or λ or ρ) which satisfies certain obvious necessary conditions can be obtained in (1) (or (2) or (3)) for some process $\{X_n\}$. Thus the efforts of various authors to weaken the mixing rates assumed in various limit theorems do lead to strictly more general theorems. Our examples are such that α, λ, and ρ converge to zero at essentially the same rate; specifically they are within constant factors of each other. This shows that ψ-mixing at a certain rate does not imply strong mixing at a faster rate.

Several of the papers referred to above deal with strictly stationary random sequences. In Section 3, we construct such a sequence which satisfies $K_1 f(k) \leq$

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