Vol. 43, No. 2 DUKE MATHEMATICAL JOURNAL© June 1976

EXTENDING FUNCTIONS FROM SUBMANIFOLDS OF THE BOUNDARY

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Introduction.

Given a smoothly bounded domain D in a complex manifold $\mathfrak{M}$, and given a smooth submanifold Γ of ∂D, it may happen that for every smooth function f on Γ, there is a holomorphic function F on D with smooth boundary values that agrees on Γ with f. We have not specified the notion of smoothness; each specification gives rise to a separate problem.

In this paper, we deal with the real analytic case. Let D be a strictly pseudoconvex domain in $\mathfrak{M}$ with $\mathcal{C}^3$ boundary. Define a real analytic submanifold Σ of $\mathfrak{M}$ that is contained in ∂D to be an *analytic interpolation manifold* (*relative to* D) if every real analytic function on Σ is the restriction to Σ of a function holomorphic on some neighborhood of $\bar{D}$. (The neighborhood depends on the function, of course.) Our main results are two characterizations of analytic interpolation manifolds:

THEOREM 1. *Σ is an analytic interpolation manifold if and only if $T_p(\Sigma) \subset T_p^{\mathbf{C}}(\partial D)$ for every $p \in \Sigma$.*

Here $T_p(\Sigma)$ is the tangent space to Σ at p, and $T_p^{\mathbf{C}}(\partial D)$ is the maximal complex subspace of $T_p(\partial D)$.

THEOREM 2. *Σ is an analytic interpolation manifold if and only if there exists a complex submanifold $\mathfrak{N}$ of a neighborhood of $\bar{D}$ such that $\mathfrak{N} \cap \bar{D} = \Sigma$.*

In order for Σ to be an analytic interpolation manifold, there should obviously be no tangential Cauchy-Riemann equations induced on Σ from $\mathfrak{M}$, *i.e.*, $T_p(\Sigma)$ should be totally real: If J is the almost complex structure on $T_p(\mathfrak{M})$, we must have $T_p(\Sigma) \cap JT_p(\Sigma) = 0$. Any Σ satisfying the differential condition of Theorem 1 is totally real and hence has real dimension not more than $m - 1$, $m = \dim_{\mathbf{C}} \mathfrak{M}$. Indeed, if $0 \neq \xi_p \in T_p(\Sigma) \cap JT_p(\Sigma)$, then there are local vector fields ξ', ξ'' on $\mathfrak{M}$ tangent to Σ along Σ, and tangent to ∂D along ∂D, with $\xi_p' = \xi_p$, $\xi_p'' = J\xi_p$. Then $[\xi', \xi'']_p \in T_p(\Sigma) \subset T_p^{\mathbf{C}}(\partial D)$. Thus, if ρ is a strictly plurisubharmonic defining function for ∂D (cf. §I below), one has

$$\begin{aligned} i\partial\bar{\partial}\rho(\xi', \xi'') &= -id(\partial\rho)(\xi', \xi'') \\ &= -i\xi'(\partial\rho(\xi'')) + i\xi''(\partial\rho(\xi')) + i\partial\rho([\xi', \xi'']), \end{aligned}$$

Received January 22, 1976. This research supported in part by NSF Grants MPS 72-05048A02 and GP38220. Some of the results of this paper were presented by the second author to the Summer Institute on Several Complex Variables, Williamstown, 1975.