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WEAK LEBESGUE SPACES AND QUANTUM MECHANICAL BINDING

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§1. Introduction.

The homogeneous Sobolev inequality gives a bound for the L^q norm of a function in terms of the L^p norm of its gradient, where $1/q = 1/p - 1/n$ and $p < n$. There are also inhomogeneous inequalities that bound the L^r norm of the function in terms of the L^p norm of the function and its gradient, where $1/p - 1/n < 1/r < 1/p$. We consider only the homogeneous inequalities for functions on Euclidean space, since the inhomogeneous inequalities are relatively straightforward.

Strichartz [11; Chap. II, Thm. 3.6] has proved an inequality that implies the homogeneous Sobolev inequality, but which is not an obvious consequence of it. The inequality is stated in terms of weak L^p spaces. His proof uses the Marcinkiewicz interpolation theorem and the constants are not made explicit. On the other hand, his treatment extends to derivatives of arbitrary—even fractional—order.

Recently Glaser, Martin, Grosse, and Thirring [6] have derived the best constants in the homogeneous Sobolev inequality and some related inequalities for the case $p = 2$ and $n = 3$. In response to a question, Martin pointed out that the methods of their paper also give a proof of the Strichartz inequality for this case.

The present paper contains a proof of the Strichartz inequality for arbitrary p, $1 \leq p < n$, and with best constants. Since interpolation is not used, the proof works even for $p = 1$. The method is an extension of that of Glaser, Martin, Grosse, and Thirring. However it should be pointed out that their paper goes much deeper, in that the best constants for the homogeneous Sobolev inequality are not obtained by the derivation from the Strichartz inequality, but require a separate variational argument.

The present treatment is based on a rearrangement lemma which uses the fact that the distribution of a function may be expressed in terms of its gradient. Thus it does not immediately extend to inequalities involving higher order derivatives. However the result for first derivatives suffices for the application to quantum mechanics in the last section.

I thank A. Martin for showing me the results of Glaser, Martin, Grosse, and Thirring which inspired this work.

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