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ON OPERATORS WITH THE DOUBLE COMMUTANT PROPERTY

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Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$, let $\mathcal{S}'$ denote its commutant, that is, $\mathcal{S}' = \{A \in \mathcal{L}(\mathcal{H}) : AS = SA \text{ for all } S \in \mathcal{S}\}$, and $\mathcal{S}'' = (\mathcal{S}')'$ its double commutant. Then clearly $\mathcal{S} \subset \mathcal{S}''$. If $\mathcal{H}$ is finite dimensional and $\mathcal{A}$ is any subalgebra of $\mathcal{L}(\mathcal{H})$, then a classical theorem in linear algebra says that $\mathcal{A} = \mathcal{A}''$. If $\mathcal{H}$ is infinite dimensional and $\mathcal{A}$ is a weakly closed *-algebra, then the von Neumann double commutant theorem says that $\mathcal{A} = \mathcal{A}''$. In general, $\mathcal{A} \neq \mathcal{A}''$ if $\mathcal{A}$ is an unstarred algebra.

For $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$ let $\mathcal{A}(\mathcal{S})$ denote the weakly closed algebra generated by $\mathcal{S}$ and the identity I. We say that $T \in \mathcal{L}(\mathcal{H})$ has the double commutant property (DCP) provided $\mathcal{A}(T) = \{T\}''$. Turner [13] has shown that algebraic operators (T is algebraic if $p(T) = 0$ for some polynomial p) have the DCP. Recently Bonsall and Rosenthal [2, Cor. 7.4] have shown that certain square roots of self-adjoint operators have the DCP. In Theorem 1 of this paper we generalize both of these results by proving that T has the DCP provided $f(T)$ is normal and has the DCP, where f is a function analytic in a neighborhood of $\sigma(T)^\wedge$ and nonconstant on components. Recall that $\sigma(T)$ denotes the spectrum of T, and that for any compact set $K \subset \mathbf{C}$, $K^\wedge$ denotes its polynomial convex hull; that is, $K^\wedge = K \cup \{\text{the bounded components of the complement of } K\}$. In this situation Theorem 2 describes $\{T\}''$ in terms of T and $\{f(T)\}''$. An essential step in the proofs of Theorems 1 and 2 is Lemma 3, a result due to Gilfeather. This lemma describes the structure of T provided $f(T)$ is normal. Finally, we give some examples and consider a slight modification of the double commutant property.

We begin by stating two results which appear in [12]. The proof of the first result follows from a theorem of Sarason [9], while the proof of the second result is straightforward.

LEMMA 1. *A normal operator N has the DCP if and only if every subspace invariant for N also reduces N.*

LEMMA 2. *Suppose $T = \sum_{n=0}^{\infty} \oplus T_n \in \mathcal{L}(\sum_{n=0}^{\infty} \oplus \mathcal{H}_n)$. If $\mathcal{A}(T) = \sum_{n=0}^{\infty} \oplus \mathcal{A}(T_n)$ and each T_n has the DCP, then T has the DCP.*

The following lemma is due to Gilfeather [5, Th 3.1]. We include a proof for completeness.

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