Vol. 43, No. 2 DUKE MATHEMATICAL JOURNAL© June 1976

THE SEVENTH COEFFICIENT OF ODD SYMMETRIC UNIVALENT FUNCTIONS

GEORGE B. LEEMAN, JR.

Let S be the collection of all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the unit disk, and denote by S_{odd} the subset of all functions

$$f(z) = z + c_3 z^3 + c_5 z^5 + c_7 z^7 + \cdots \tag{1}$$

in S. It is well known that $|c_3| \leq 1$ for such functions, and Littlewood and Paley [8] proved the existence of a constant M such that $|c_{2n-1}| \leq M$ for all f in (1), $n = 2, 3, \cdots$. Levin [7] showed that one can take $M = 3.39$, Kung Sun [5] obtained the better estimate $M = 2.54$, and Milin [11] improved the result to $M = 1.17$, but this value is not sharp. It had been conjectured in [8] that the best value for M is 1, for this fact would easily imply the Bieberbach conjecture. However, for each $n > 2$ Schaeffer and Spencer [12] constructed functions of the form (1) with real coefficients such that $c_{2n-1} > 1$.

In 1933 Fekete and Szegö [2] found the precise bound

$$|c_5| \leq \tfrac{1}{2} + e^{-2/3}$$

for functions of the form (1). To the best of our knowledge no sharp estimates have been found since that time, even for the subcollection of functions in S_{odd} with real coefficients. In this paper we shall determine the precise bound for c_7 in (1) within this subclass. In addition we will identify all extremal functions.

Before beginning, let us sketch our approach. We first convert our task to maximizing over S a functional in the coefficients a_2, a_3, and a_4. Now Charzynski and Schiffer [1] gave a beautiful proof of the Bieberbach conjecture for the fourth coefficient by exhibiting an inequality involving the same three coefficients. By using their result we reduce our problem to considering a function $k(u, v)$ of two variables; we must maximize k over the collection R of all (u, v) for which there exists a function $z \to z + uz^2 + vz^3 + \cdots$ in S with real coefficients. However, the work of Schaeffer and Spencer [13] and Jenkins [4] shows that the set R is extremely complicated. Hence we work with a slightly larger set Ω where it is easy to find that point (u_0, v_0) at which k is maximized. Sharpness of our result depends on choosing Ω carefully enough so that (u_0, v_0) is not in $\Omega - R$. To examine the case of equality we determine the relevant quadratic differential and construct all functions associated with it. Then only a simple numerical computation is required to find the function with the desired properties.

Received May 19, 1975.