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COVERING PROPERTIES OF RANDOM SERIES

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Let E be a closed subset of [0, 1], of Hausdorff dimension α, $0 < \alpha < 1$; by a classical theorem of Frostman [1], E can be mapped onto a set of positive Lebesgue measure by a function f of class $\wedge^{\beta}$ if $0 < \beta < \alpha$, and plainly this fails if $\alpha < \beta$. In the case $\beta < \alpha$, the function f depends very strongly on E, and the usual method of constructing f from E leads to the suspicion that the structure of E must be reflected in some oscillation in f. The theorem to be proved goes in the opposite direction.

THEOREM. *To each pair of numbers α, β in the region $0 < \beta < \alpha < 1$, there exist some n function f_i of class $\wedge^{\beta}[0, 1]$ with this property: for each closed set $E \subseteq [0, 1]$ of dimension $> \alpha$, almost all functions $f_y = \sum y_i f_i$ ($y_i \in R$) transform E onto a linear set with non-void interior. (Here $n > 8(\alpha - \beta)^{-2}$ seems to be large enough).*

1. In the proof of the theorem we need an auxiliary construction of measures. Let μ be a probability measure on a compact subset of Euclidean space R^n and let $N \geq 1$. There is then defined the function

$$\sum_1^{2N} u_j x_j \,, \quad \text{for} \quad x_j \in R^n, \quad 1 \leq |u_j| \leq 2 \; (1 \leq j \leq 2N).$$

LEMMA. *Suppose that the set*

$$\left\{ \left\| \sum_1^{2N} u_j x_j \right\| < \lambda \right\}$$

has measure $\ll \lambda^{2N+3}$, in the product measure $du_1 \cdots du_{2N} \mu^{(2N)}$ on $R^{2N} \times R^{2N}$, uniformly with respect to λ in (0, 1).

Then for almost all y in R^n (with respect to Lebesgue measure) the distribution of the variable $y \cdot x$, with respect to μ, has a continuous derivative.

Proof. Let $\sigma = \sigma(y)$ be the distribution of the variable $y \cdot x$, so the Fourier-Stieltjes transform of $\sigma(y)$ is represented by the formula

$$\hat{\sigma}(y, s) = \int \exp -2\pi i s (y \cdot x) \mu(dx), \qquad -\infty < s < \infty.$$

Then $\sigma(y)$ has a continuous density if there is an $\eta > 0$ so that

$$\left| \int_{T_k} \hat{\sigma}(y, s) \exp 2\pi i s \xi \, ds \right| \ll 2^{-\eta k}$$

uniformly for bounded sets on the ξ-axis, where $T_k = (2^k \leq |s| \leq 2^{k+1})$.

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