Vol. 43, No. 2 DUKE MATHEMATICAL JOURNAL© June 1976

THE COMMUTATIVITY OF TUNNEL SUM

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Introduction. This paper studies the local knot cobordism group $C(M, \xi)$ of embeddings of a compact manifold M in the total space $E(\xi)$ of a 2-plane bundle over M. For M a closed manifold, Cappell and Shaneson defined the set $C(M, \xi)$ and showed that it admits a geometrically defined "tunnel sum" group operation. The aim of this paper is to provide a geometric proof of [CS, Cor. 5.3]:

THEOREM. *The tunnel sum group operation in $C(M, \xi)$ is commutative.*

Let $\mathfrak{F} : Z[\pi_1(\dot{E}(\xi))] \to Z[\pi_1(M)]$ be the homomorphism of integral group rings induced by the projection map of the boundary circle bundle of ξ. The commutative square

$$\Phi = \begin{array}{ccc} Z[\pi'] & \xrightarrow{id} & Z[\pi'] \\ \downarrow id & & \downarrow \mathfrak{F} \\ Z[\pi'] & \xrightarrow{\mathfrak{F}} & Z[\pi] \end{array} \qquad \begin{array}{l} \pi' = \pi_1(\dot{E}(\xi)) \\ \\ \pi = \pi_1(M) \end{array}$$

gives rise to a series of abelian algebraic K-theoretic functors $\Gamma_i(\Phi)$, defined and exploited in [CS]. The abelian groups $\Gamma_i(\Phi)$ are obstruction groups for codimension two surgery problems. The above theorem was proved in [CS] by exhibiting an injective homomorphism $\Sigma : C(M, \xi) \to \Gamma_{k+3}(\Phi)$, where M has dimension k.

In this paper, we first embed $C(M, \xi)$ as a subset of a "fake" local knot cobordism set $\tilde{C}(M, \xi)$ of embeddings in $E(\xi)$ of manifolds homotopy equivalent to M. Our main technical result is:

THEOREM 1. *Let ξ be a 2-plane bundle over a compact manifold M of dimension $k \geq 4$. Then there is a natural bijection $\tilde{\Sigma} : \tilde{C}(M, \xi) \to \Gamma_{k+3}(\Phi)$.*

In order to prove geometrically the commutativity of tunnel sum we observe that there is a codimension zero submanifold $X \times I$ of M and a natural bijection $j : \tilde{C}(X \times I) \to \tilde{C}(M)$ which commutes with $\tilde{\Sigma}$ above. Thus M may be viewed as a near suspension of the codimension one submanifold X. In an obvious way $\tilde{C}(X \times I)$ admits a "stacked sum" group operation. By iterating the desuspension idea, we obtain a stacked sum isomorphism $j : \tilde{C}(Y \times I \times I) \to \tilde{C}(X \times I)$. Since stacking in $\tilde{C}(Y \times I \times I)$ may be accomplished in two different ways, standard arguments show that the stacked sum operation in $\tilde{C}(Y \times I \times I)$, hence in $\tilde{C}(X \times I)$, is abelian. Using Theorem 1 above and

Received December 19, 1975. Research partially supported by a CUNY-FRAP grant.