Vol. 43, No. 2 DUKE MATHEMATICAL JOURNAL© June 1976

JULIA'S LEMMA FOR FUNCTION ALGEBRAS

I. GLICKSBERG

§1. Various results of function theory hold more generally for uniform algebras. Indeed, many classical arguments extend directly to the broader context because of the validity of certain basic analogues, notably Rossi's local maximum modulus theorem [3], the existence of Jensen measures [3], and a simple fact concerning the vanishing of function algebra elements corresponding to the classical vanishing of an analytic function zero on an open subset of the boundary of a domain [3, 4]. Among such results is an analogue of Schwarz's lemma [4, 4.1] and our purpose here is to point out how just the classical argument applies to yield an analogue of Julia's lemma from this, which, in turn, can be applied to give some criteria insuring that one element of our algebra is an analytic function of another. (These have been greatly simplified by some comments of John Wermer, to whom the author is much indebted.)

Let $1 \in A$, a closed separating subalgebra of $C(M)$, where M is a compact Hausdorff space, and let $\partial \subset M$ be a closed boundary, i.e., a superset of the Šilov boundary ∂_A . We *shall assume local maximum modulus holds relative to* ∂ in the sense that for each open $U \subset M\backslash\partial$ we have

$$|a| \leq \sup |a(\partial U)| \text{ on } U$$

for all $a \in A$; that this is the case for $M = M_A$, the spectrum of A, is the content of Rossi's local maximum modulus theorem. Our abstract Schwarz lemma is a consequence of the validity of local maximum modulus and asserts [4, 4.1] that *if f, $g \in A$ and f/g is bounded on $M\backslash g^{-1}(0)$, then*

$$\sup \left|\frac{f}{g}(M\backslash g^{-1}(0))\right| = \sup \left|\frac{f}{g}(\partial\backslash g^{-1}(0))\right|. \tag{1.1}$$

(For the classical case, where A is the disc algebra, $g(z) \equiv z$.)

§2. Julia's lemma arises from the convergence of non-euclidean discs in the open unit disc D° as their (non-euclidean) centers tend to 1 while their radii behave appropriately. We shall follow Ahlfors [1] in deriving our version. (D is the closed disc.)

We first rewrite the defining inequality for the non-euclidean disc

$$K(a, R) = \left\{z \in D^\circ : \left|\frac{z - a}{1 - \bar{a}z}\right| < R\right\}$$

Received November 13, 1975. Revision received February 12, 1976. Work supported in part by the National Science Foundation.