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CONVERGENCE OF REVERSED MARTINGALES WITH MULTIDIMENSIONAL INDICES

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1. Introduction.

Let $d \geq 2$ be an integer and let Z_+^d denote the positive d-dimensional integer lattice points. For k, $1 \leq k \leq d$, let $(\Omega_k, \mathcal{A}_k, P_k)$ be a probability space, and set $\Omega = \prod_k \Omega_k$, $\mathcal{A} = \bigotimes \mathcal{A}_k$ and $P = \prod_k P_k$. An indexed random variable, defined on $(\Omega, \mathcal{A}, P)$, will always be interpreted as having its index in Z_+^d (unless explicitly otherwise stated). The notation $\mathbf{m} < \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \cdots, m_d)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_d) \in Z_+^d$, means that $m_i \leq n_i$, $i = 1, 2, \cdots, d$, (cf. [3]) and $\mathbf{n} \to \infty$ is to be understood as $n_i \to \infty$, $i = 1, 2, \cdots, d$. Also, $|\mathbf{n}|$ is used to denote $\prod_{k=1}^d n_k$.

Let $\{\mathfrak{F}_\mathbf{n} = \mathfrak{F}_{n_1}^{(1)} \otimes \mathfrak{F}_{n_2}^{(2)} \otimes \cdots \otimes \mathfrak{F}_{n_d}^{(d)}; \mathbf{n} \in Z_+^d\}$ be a sequence of σ-algebras contained in $\mathcal{A}$ and such that $\mathfrak{F}_\mathbf{m} \subset \mathfrak{F}_\mathbf{n}$ if $\mathbf{m} < \mathbf{n}$. Following Cairoli [1] and Cairoli and Walsh [3] we define a martingale to be $\{X_\mathbf{n}, \mathfrak{F}_\mathbf{n}; \mathbf{n} \in Z_+^d\}$, where

$$(1.1) \qquad X_\mathbf{n} \text{ is } \mathfrak{F}_\mathbf{n}\text{-measurable and integrable for every } \mathbf{n};$$

$$(1.2) \qquad X_\mathbf{m} = E(X_\mathbf{n} \mid \mathfrak{F}_\mathbf{m}) \quad \text{a.s.} \quad \text{if} \quad \mathbf{m} < \mathbf{n}.$$

As in the one-dimensional case it is easy to see that (1.2) is equivalent to

$$(1.3) \qquad \int_\Lambda X_\mathbf{m}\, dP = \int_\Lambda X_\mathbf{n}\, dP; \qquad \text{for} \quad \Lambda \in \mathfrak{F}_\mathbf{m}, \qquad \mathbf{m} < \mathbf{n}.$$

A submartingale will be defined exactly as a martingale except that the equalities in (1.2) and (1.3) are changed to "$\leq$".

If $\{\mathfrak{F}_\mathbf{n}\}$ is a sequence of σ-algebras of product type contained in $\mathcal{A}$, such that $\mathfrak{F}_\mathbf{m} \supset \mathfrak{F}_\mathbf{n}$ if $\mathbf{m} < \mathbf{n}$, we define a reversed martingale to be $\{X_\mathbf{n}, \mathfrak{F}_\mathbf{n}; \mathbf{n} \in Z_+^d\}$, where (1.1) is satisfied and

$$(1.4) \qquad X_\mathbf{m} = E(X_\mathbf{n} \mid \mathfrak{F}_\mathbf{m}) \quad \text{a.s.} \quad \text{if} \quad \mathbf{n} < \mathbf{m}.$$

Again it is easy to see that (1.4) is equivalent to

$$(1.5) \qquad \int_\Lambda X_\mathbf{m}\, dP = \int_\Lambda X_\mathbf{n}\, dP \quad \text{for} \quad \Lambda \in \mathfrak{F}_\mathbf{m}, \qquad \mathbf{n} < \mathbf{m}.$$

A reversed submartingale will be defined the same way except that the equalities in (1.4) and (1.5) are changed to "$\leq$".

Also, in the reversed cases we set $\mathfrak{F} = \bigcap_\mathbf{n} \mathfrak{F}_\mathbf{n}$.

In [1] maximal inequalities corresponding to Doob [5], 314 and 317 are proved

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