Vol. 43, No. 2 DUKE MATHEMATICAL JOURNAL© June 1976

SPHERICAL REARRANGEMENTS, SUBHARMONIC FUNCTIONS, AND *-FUNCTIONS IN n-SPACE

ALBERT BAERNSTEIN II

B. A. TAYLOR

1. Introduction.

Suppose that u is a real-valued function defined in a plane annulus $A = \{z : R_1 < |z| < R_2\}$ which is integrable on each circle $|z| = r$, $R_1 < r < R_2$. The **-function* of u is the function u^* defined in A^+, the open upper half of A, by

$$u^*(re^{i\theta}) = \sup_E \int_E u(re^{it})\, dt \qquad (R_1 < r < R_2\,,\ 0 < \theta < \pi),$$

where the sup is taken over all sets $E \subset [-\pi, \pi]$ with Lebesgue measure 2θ. Alternatively,

$$u^*(re^{i\theta}) = \int_{-\theta}^{\theta} \tilde{u}(re^{it})\, dt$$

where $\tilde{u}$ is the function obtained from u by symmetric decreasing rearrangement on each circle. (cf. [4], Section 3).

The first author has shown ([4], Theorem A) that u^* enjoys the following property.

THEOREM A. *If u is subharmonic in A, then u^* is subharmonic in A^+.*

This theorem has been the main tool in the solution of certain extremal problems involving univalent functions, Nevanlinna theory, and other branches of function theory. See [2], [3], [4], [5].

The present study grew out of an attempt to find an analogue of Theorem A for subharmonic functions in higher dimensions. It will be convenient to denote the dimension by $p + 2$. In the Euclidean space $\mathbf{R}^{p+2}$, $p \geq 1$, we consider for a point x the spherical coordinates r, θ defined by $r = |x|$, $x_1 = r \cos \theta$, where $x = (x_1, \cdots, x_{p+2})$ and $|x|^2 = \sum_{i=1}^{p+2} x_i^2$. (We will not need to consider the other angular coordinates). Let $\mathbf{S}^{p+1}$ denote the unit sphere $\{x \in \mathbf{R}^{p+2} : |x| = 1\}$, $C(\theta_0)$ the spherical cap on $\mathbf{S}^{p+1}$ given by $C(\theta_0) = \{x \in \mathbf{S}^{p+1} : 0 \leq \theta < \theta_0\}$ and $d\sigma$ the surface area measure on $\mathbf{S}^{p+1}$. Suppose that u is a real-valued function defined in the spherical shell

$$A(R_1, R_2) = \{x \in \mathbf{R}^{p+2} : R_1 < |x| < R_2\}$$

Received December 26, 1975. This research partially supported by National Science Foundation Grant MPS73-08854 A02 (formerly GP-38959) (Baernstein) and GP-37628 (Taylor)