Vol. 43, No. 2 DUKE MATHEMATICAL JOURNAL© June 1976

SELF-ADJOINT OPERATORS AND FORMALLY REAL FIELDS

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Few theorems are more familiar than the diagonalizability of self-adjoint operators on a finite-dimensional inner product space over the reals. The result automatically extends to real closed fields, but apart from that little seems to have been known about its range of validity. We here give necessary and sufficient conditions for the theorem to be true, showing in particular that it holds over all hereditarily euclidean fields. For such fields we go on to show how even for an indefinite inner product the index gives a bound on the degrees of elementary divisors of self-adjoint operators.

1. Separability of normal operators.

PROPOSITION 1. *Let k be a field,* char $(k) \neq 2$. *Let B be a symmetric bilinear form on a finite-dimensional k-space V, and assume the quadratic form $Q(v) = B(v, v)$ is anisotropic. Let $S : V \to V$ be a linear map which is normal with respect to B, i.e., commutes with its adjoint. Then S is separable.*

Proof. If S is normal and $Sv = 0$, then $0 = B(Sv, Sv) = B(S^*Sv, v) = B(SS^*v, v) = B(S^*v, S^*v)$, so $S^*v = 0$ since Q is anisotropic. Thus from $S(Sv) = 0$ we can deduce $S^*Sv = 0$, whence $0 = B(S^*Sv, v) = B(Sv, Sv)$ and $Sv = 0$. Hence S cannot be nilpotent without being zero. Let f now be the product of the distinct irreducible factors in the minimal polynomial of a normal operator S. Then $f(S)$ is nilpotent; and $f(S)^* = f(S^*)$ commutes with $f(S)$, since S and S^* commute. Hence $f(S)$ is zero, and S is semisimple.

If S is not separable, there is some finite purely inseparable extension L of k such that S extended to $V \otimes_k L$ is not semisimple. Clearly the extended S is normal with respect to the extension of B. But by a theorem of Springer [2, p. 198] the extended quadratic form is still anisotropic, since the degree $|L : k|$ is odd. This we have seen is impossible, and so S must be separable. □

The first part of this reasoning is of course familiar; the proposition is included here because it is as close as one can come to diagonalizability without assuming special properties of k. It is worth noting that the result always fails when Q is (nondegenerate but) isotropic. Indeed, we can then split off an orthogonal summand with basis e_1, e_2 where $Q(xe_1 + ye_2) = xy$; if we set $S(e_1) = e_2$ and $S(e_2) = 0$, with $S = 0$ on the orthogonal complement, then S will be self-adjoint and nilpotent.

Received December 19, 1975. This work was supported in part by an N.S.F. contract at Cornell University.