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ON THE CLASSIFICATION OF LAGRANGE IMMERSIONS

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In this note, we give a homotopy-theoretic classification of the lagrange immersions $\Lambda \to M$ of a smooth n-manifold Λ into a smooth symplectic $2n$-manifold M. Recall that a symplectic structure σ on M is a closed nondegenerate 2-form and that a smooth immersion $\lambda : \Lambda \to M$ is called a lagrange immersion if σ vanishes on each pair of vectors in TM tangent to $\lambda\Lambda$, that is, if $\lambda^* \sigma = 0$.

To each lagrange immersion λ we can associate its differential $d\lambda : T\Lambda \to$ TM. By definition, $d\lambda$ is a bundle map which takes each fiber Λ_p to a lagrangian plane in $M_{\lambda p}$, that is, an n-plane on which σ vanishes. We will call such bundle maps l-bundle maps, so that d sends each lagrange immersion to an l-bundle map.

We will say that two lagrange immersions λ_0 and λ_1 are l-regularly homotopic if there is a smooth regular homotopy λ_t between λ_0 and λ_1, such that λ_t is a lagrange immersion for each t. Similarly, we can speak of a homotopy through l-bundle maps of $T\Lambda$ in TM.

A bundle map $\Phi : T\Lambda \to$ TM will be called admissible if it covers a map $\phi : \Lambda \to M$ such that the cohomology class of $\phi^*\sigma$ vanishes in $H^2(\Lambda; R)$. In particular, if λ is a lagrange immersion, $d\lambda$ is admissible.

THEOREM 1. *d induces a $1 - 1$ correspondence between l-regular homotopy classes of lagrange immersions $\Lambda \to M$ and homotopy classes of admissible l-bundle maps $T\Lambda \to TM$.*

A version of Theorem 1 has been announced by M. Gromov [G]. Our approach is inspired by Haefliger and Poenaru's proof of the immersion theorem for piecewise linear manifolds [HP]. We thank Professor Gromov for many helpful suggestions.

Using some standard results in bundle theory, we can restate our result as

THEOREM 2. *Suppose $f : \Lambda \to M$ is given ,with $f^*\sigma$ cohomologically trivial. Then the l-regular homotopy classes of lagrange immersions $\Lambda \to M$ homotopic to f are in $1 - 1$ correspondence with homotopy classes of sections of a bundle over Λ with fiber $O(n) \times U(n)/O(n)$.*

Theorem 1 leads to an obstruction theory for deformations of lagrange immersions $\lambda : \Lambda \to M$. To describe this, it is convenient to introduce the l-distributions $\Lambda \to$ TM which send each point of Λ to a lagrangian plane. Those which cover a given map $\phi : \Lambda \to M$ can be viewed as sections of a bundle ($\phi^*l(M)$, see 2.1) over Λ with fibre $U(n)/O(n)$. Thus, the obstructions to finding

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