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NONLINEAR CAUCHY–RIEMANN EQUATIONS AND q-PSEUDOCONVEXITY

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0. Introduction. This paper is devoted to a study of the equation

$$\bar{\partial} f \wedge (\partial\bar{\partial} f)^q = 0, \tag{1}$$

where f is a smooth function on a complex (analytic) manifold Ω and q is a nonnegative integer. If Ω has dimension n and if $0 < q < n$, then (1) is equivalent to a system of nonlinear partial differential equations which may be regarded as a generalization of the usual Cauchy–Riemann equations. Our main result (Theorem 3 in Section 3) is that the solutions of (1) define a notion of convexity which is related (at least locally) to q-pseudo convexity in the same way that holomorphic convexity is related to ordinary pseudoconvexity.

1. Definition and some basic properties of q-holomorphic functions.

Definition. Let Ω be a complex manifold. Define

$$\mathcal{O}_q(\Omega) = \{f \in C^\infty(\Omega) \mid \bar{\partial} f \wedge (\partial\bar{\partial} f)^q = 0\}.$$

(Note that $\mathcal{O}_0(\Omega) \subseteq \mathcal{O}_1(\Omega) \subseteq \cdots \subseteq \mathcal{O}_n(\Omega) = \mathcal{O}_{n+1}(\Omega) = \cdots = C^\infty(\Omega)$, if Ω has dimension n.) If $f \in \mathcal{O}_q(\Omega)$, we will say that f is q-holomorphic on Ω.

Before describing some examples of q-holomorphic functions it is convenient to develop some additional criteria for recognizing them.

PROPOSITION 1. *Let $\phi : \Omega_0 \to \Omega_1$, where Ω_0, Ω_1 are complex manifolds and ϕ is holomorphic. If $f \in \mathcal{O}_q(\Omega_1)$, then $f \circ \phi \in \mathcal{O}_q(\Omega_0)$.*

Proof. $\bar{\partial}(f \circ \phi) \wedge (\partial\bar{\partial}[f \circ \phi])^q = (\bar{\partial} f \wedge [\partial\bar{\partial} f]^q) \circ \phi = 0.$

Corollary. Let Ω be a complex manifold and let V be a smooth subvariety of Ω. If $f \in \mathcal{O}_q(\Omega)$, then $f|_V \in \mathcal{O}_q(V)$.

Proof. Apply Proposition 1 with ϕ the inclusion map of V into Ω.

Definition. Let Ω be an open subset of $\mathbf{C}^n$, and let $z = \{z_1, \cdots, z_n\}$ be coordinates for $\mathbf{C}^n$. If $f \in C^\infty(\Omega)$ and $x \in \Omega$ we define

$$M_x^z(f) = \begin{pmatrix} f_{\bar{z}_1}(x) & \cdots & f_{\bar{z}_n}(x) \\ f_{z_1\bar{z}_1}(x) & \cdots & f_{z_1\bar{z}_n}(x) \\ \vdots & & \vdots \\ f_{z_n\bar{z}_1}(x) & \cdots & f_{z_n\bar{z}_n}(x) \end{pmatrix}.$$

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