NONLINEAR CAUCHY–RIEMANN EQUATIONS AND q-PSEUDOCONVEXITY

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0. Introduction. This paper is devoted to a study of the equation

\[ \partial f \wedge (\partial \bar{\partial} f)^q = 0, \]

where \( f \) is a smooth function on a complex (analytic) manifold \( \Omega \) and \( q \) is a nonnegative integer. If \( \Omega \) has dimension \( n \) and if \( 0 < q < n \), then (1) is equivalent to a system of nonlinear partial differential equations which may be regarded as a generalization of the usual Cauchy–Riemann equations. Our main result (Theorem 3 in Section 3) is that the solutions of (1) define a notion of convexity which is related (at least locally) to \( q \)-pseudo convexity in the same way that holomorphic convexity is related to ordinary pseudoconvexity.

1. Definition and some basic properties of \( q \)-holomorphic functions.

Definition. Let \( \Omega \) be a complex manifold. Define

\[ \mathcal{O}_q(\Omega) = \{ f \in C^\infty(\Omega) \mid \partial f \wedge (\partial \bar{\partial} f)^q = 0 \}. \]

(Note that \( \mathcal{O}_0(\Omega) \subseteq \mathcal{O}_1(\Omega) \subseteq \cdots \subseteq \mathcal{O}_n(\Omega) = \mathcal{O}_{n+1}(\Omega) = \cdots = C^\infty(\Omega) \), if \( \Omega \) has dimension \( n \).) If \( f \in \mathcal{O}_q(\Omega) \), we will say that \( f \) is \( q \)-holomorphic on \( \Omega \).

Before describing some examples of \( q \)-holomorphic functions it is convenient to develop some additional criteria for recognizing them.

Proposition 1. Let \( \phi : \Omega_0 \to \Omega_1 \), where \( \Omega_0 \), \( \Omega_1 \) are complex manifolds and \( \phi \) is holomorphic. If \( f \in \mathcal{O}_q(\Omega_1) \), then \( f \circ \phi \in \mathcal{O}_q(\Omega_0) \).

Proof. \( \partial(f \circ \phi) \wedge (\partial \bar{\partial} (f \circ \phi))^q = (\partial f \wedge [\partial \bar{\partial} f]^q) \circ \phi = 0. \)

Corollary. Let \( \Omega \) be a complex manifold and let \( V \) be a smooth subvariety of \( \Omega \). If \( f \in \mathcal{O}_q(\Omega) \), then \( f|_V \in \mathcal{O}_q(\Omega) \).

Proof. Apply Proposition 1 with \( \phi \) the inclusion map of \( V \) into \( \Omega \).

Definition. Let \( \Omega \) be an open subset of \( \mathbb{C}^n \), and let \( z = \{ z_1, \cdots, z_n \} \) be coordinates for \( \mathbb{C}^n \). If \( f \in C^\infty(\Omega) \) and \( x \in \Omega \) we define

\[ M_x(f) = \begin{pmatrix} f_{z_1}(x) & \cdots & f_{z_n}(x) \\ f_{z_1\bar{z}_1}(x) & \cdots & f_{z_n\bar{z}_n}(x) \\ \vdots & \vdots & \vdots \\ f_{z_n\bar{z}_1}(x) & \cdots & f_{z_n\bar{z}_n}(x) \end{pmatrix}. \]

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