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HOMOLOGICAL DIMENSIONS OF THE ISOTROPY RING

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§1. Introduction.

In Willson [7] a ring $\mathcal{I}$ called the isotropy ring was introduced to treat relationships between equivariant homology theories, especially in situations where there were many different isotropy subgroups. The major result was a spectral sequence with $E_{p,q}^2 = \mathrm{Tor}_p^{\mathcal{I}}(A_q, M)$, where A_q and M were certain right and left $\mathcal{I}$-modules. This spectral sequence has been utilized in Willson [8] to compute the homology with Z_{p^r} coefficients of the orbit spaces of Z_{p^s} actions on spheres.

In this paper, we study algebraic properties of the ring $\mathcal{I}$. We concentrate our attention on various homological dimensions of $\mathcal{I}$. In particular we shall prove that under fairly general hypotheses, $\mathcal{I}$ has finite left global dimension. This result is of significance since it indicates the complexity of the spectral sequences which may arise as above. In some cases, our methods give easily computable upper bounds on these dimensions.

In §2 we give a brief description of $\mathcal{I}$. In §3 we study the radical of $\mathcal{I}$. In §4 we relate the various homological dimensions of $\mathcal{I}$. In §5 we prove our major theorem, Theorem 5.5. In §6 we indicate some practical methods for obtaining upper bounds on the homological dimensions, and we compute some examples.

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§2. The isotropy ring $\mathcal{I}$.

In this section we define the ring $\mathcal{I}$ and summarize some of its most basic properties. For a fuller treatment, the reader should see Willson [7, §4].

Definition. Let G be a topological group and let $\mathcal{H} = \{H_1, H_2, \cdots\}$ be a collection of closed subgroups of G satisfying that if $i \neq j$, then H_i and H_j are not conjugate subgroups in G. We call $\mathcal{H}$ a *list of isotropy groups* or, more simply, a *list.* If $H_i, H_j \in \mathcal{H}$, we denote by $\mathfrak{M}(H_j, H_i)$ the set of G-homotopy classes of G-maps from G/H_i to G/H_j. (We denote by G/H the collection of left cosets of H by elements of G; a G-map $f : G/H_i \to G/H_j$ is a map so $f(g \cdot aH_i) = gf(aH_i)$ for all $g, a \in G$.)

Let F be a commutative ring, G a topological group, $\mathcal{H}$ a list. Then the *isotropy ring* $\mathcal{I}_G^{F,\mathcal{H}}$ is the free F-module on $\cup_{H,K\in\mathcal{H}} \mathfrak{M}(H, K)$. A ring structure is imposed on $\mathcal{I}$ by composition of G-maps. Explicitly, let $\varphi : G/H \to G/J$, $\psi : G/K \to G/L$ be G-maps, a and $b \in F$. We define $(a\varphi)(b\psi)$ to equal 0 if

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