Vol. 43, No. 1 DUKE MATHEMATICAL JOURNAL© March 1976

ON THE MOD p EIGENVALUES OF HECKE OPERATORS

ALAN ADOLPHSON

1. Introduction.

Let p be an odd prime, let $\mathbf{F}_p$ denote the prime field of characteristic p, and let $\bar{\mathbf{F}}_p$ be the algebraic closure of $\mathbf{F}_p$. Let $A(\lambda) \in \mathbf{F}_p[\lambda]$ be the Hasse invariant:

$$A(\lambda) = (-1)^{(p-1)/2} \sum_{j=1}^{(p-1)/2} ((\tfrac{1}{2})_j/j!)^2 \lambda^j . \tag{1}$$

Let ψ be the linear operator on $\bar{\mathbf{F}}_p[\lambda]$ defined by linearity and the condition

$$\psi(\lambda^n) = \begin{cases} \lambda^{n/p} & \text{if } p \mid n \\ 0 & \text{if } p \nmid n. \end{cases}$$

For k a non-negative even integer, consider the endomorphism $\psi \circ A^k$ of $\bar{\mathbf{F}}_p[\lambda]$ (multiplication by A^k followed by ψ). By Dwork [2], the characteristic polynomial $\det(I - t(\psi \circ A^k))$ on $\bar{\mathbf{F}}_p[\lambda]$ can be computed by restricting to $V_{k/2}$ (where V_n denotes the vector space of polynomials of degree $\leq n$, for $n = 0, 1, 2, \cdots$).

For k even, $k \geq 4$, let $T_{k+2}(p)$ be the p-th Hecke operator acting on $S_{k+2}(\Gamma(2))$, the space of cusp forms of weight $k + 2$ with respect to $\Gamma(2)$ (= principal congruence subgroup of level 2). Note that $\det(I - tT_{k+2}(p) \mid S_{k+2}(\Gamma(2)))$ has integral coefficients.

The following result is well known (see [1], [3], [4, Eq. A3.3.3]):

THEOREM 1. *For k even, $k \geq 4$,*
$\det(I - tT_{k+2}(p)) \equiv \det(I - t(\psi \circ A^k) \mid V_{k/2})/(1 - t)^2 \pmod p.$

This article consists of applications of Theorem 1. By studying the operator $\psi \circ A^k$ on $V_{k/2}$ we will obtain information on the mod p eigenvalues of $T_{k+2}(p)$. In the next two sections we give new proofs of some well-known results, including a simplified proof of a theorem of Dwork [3]. In the last section we prove a sort of duality theorem for the (p-adic) non-unit eigenvalues. We are indebted to B. Dwork for many helpful discussions of these questions.

2. Consequences of theorem 1.

For other proofs of the results of this section, see Serre [5].

COROLLARY 1. *For k even, $k \geq 4$,*

$$\det(I - tT_{k+2}(p)) \equiv \det(I - tT_{k+p+1}(p)) \pmod p,$$

i.e., the mod p *eigenvalues are periodic of period $p - 1$ with respect to the weight.*

Received August 11, 1975. Research partially supported by an N.S.F. Grant.