Vol. 43, No. 1 DUKE MATHEMATICAL JOURNAL© March 1976

GENERALIZED PROJECTIONS AND REDUCIBLE SUBNORMAL OPERATORS

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1. If A is a bounded operator on a Hilbert space, its spectrum and point spectrum will be denoted by $\sigma(A)$ and $\sigma_p(A)$ respectively. An operator T on a Hilbert space $\mathfrak{H}$ is said to be subnormal if it has a normal extension on a Hilbert space $\mathfrak{K} \supset \mathfrak{H}$. Throughout the sequel, the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$ will be denoted by P. Concerning subnormal operators, see Halmos [2]. We recall some properties. A subnormal operator T has a minimal normal extension N and $\sigma(N) \subset \sigma(T)$ (P. R. Halmos); further, $\sigma(T)$ consists of $\sigma(N)$ together with some of the holes of $\sigma(N)$ (J. Bram). A subnormal T is called completely subnormal if it has no normal part, that is, if there exists no non-trivial subspace of $\mathfrak{H}$ which reduces T and on which T is normal. If T is subnormal and if $z \in \sigma_p(T)$ then $\bar{z} \in \sigma_p(T^*)$ and, if T is completely subnormal, $\sigma_p(T)$ is empty. If X is a compact set of the complex plane, let $C(X)$ and $R(X)$ denote respectively the continuous functions on X and the functions uniformly approximable on X by rational functions with poles off X. It was shown by Clancey and Putnam [1] that X is the spectrum of a completely subnormal operator if and only if $R(X \cap D^-) \neq C(X \cap D^-)$ whenever D is an open disk intersecting X in a non-empty set.

If A is any bounded operator and if C is any (positively oriented) rectifiable simple closed curve lying outside $\sigma(A)$, then one can define the Riesz integral

$$L = -(2\pi i)^{-1} \int_C (A - t)^{-1}\, dt \tag{1.1}$$

as a (bounded) operator satisfying the projection property $L^2 = L$. In case $\sigma = \sigma(A) \cap \operatorname{int} C \neq \emptyset$, then the space $\mathfrak{M} = L(\mathfrak{H})$ is invariant under A and $\sigma(A \mid \mathfrak{M}) = \sigma$. See Riesz and Sz.-Nagy [7], p. 418.

In certain cases, L is self-adjoint and hence is an orthogonal projection. This occurs, for example, if A is normal. More generally, let A be subnormal and let $\sigma(A)$ be the union of two non-empty disjoint parts σ_1 and σ_2. If C is a rectifiable simple closed curve lying outside $\sigma(A)$ and separating σ_1 and σ_2, then L of (1.1) is an orthogonal projection and A has the direct sum representation $A = A_1 \oplus A_2$ on $\mathfrak{H} = L(\mathfrak{H}) \otimes (\mathfrak{H} \ominus L(\mathfrak{H}))$ and $\sigma(A_k) = \sigma_k$ $(k = 1, 2)$. See Williams [9], pp. 97–98.

It is noteworthy that an analogue of the above result does not hold in general for hyponormal operators. (An operator A is hyponormal if $A^*A - AA^* \geq 0$.

Received July 22, 1975. Revision received October 27, 1975. This work was supported by a National Science Foundation research grant.