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SOLUTIONS OF THE KORTEWEG-DE VRIES EQUATION IN FRACTIONAL ORDER SOBOLEV SPACES

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1. Introduction.

In Bona and Smith [3], it is shown that for k an integer larger than 1, the pure initial-value problem for the Korteweg-de Vries equation

$$(1.1) \qquad u_t + uu_x + u_{xxx} = 0, \qquad u(x, 0) = g(x), \qquad x \in \mathbf{R}, \qquad t \geq 0,$$

has a unique solution u in $C_b(0, \infty; H^k)$ corresponding to initial data g in the Sobolev space H^k. Recently J.-C. Saut [10] has extended this result to non-integral values of k using a non-linear interpolation theorem of Tartar [11]. He showed that if $r > 3$, $\mu = [r] + 1 - r$ and $g \in H^{r+\frac{1}{3}\mu+\epsilon}$ for some $\epsilon > 0$, then for each $T > 0$, u lies in $L^\infty(0, T; H^r)$. For the case $2 < r < 3$, Saut has the slightly weaker result that if $g \in H^{r+\frac{1}{2}\mu}$, then for each $T > 0$, u lies in $L^\infty(0, T; H^r)$. Thus it would seem that some spatial regularity is lost in solving (1.1) for initial data in non-integral Sobolev classes. The purpose of this note is to show that this is not the case, namely, that for data in H^s, $s \geq 2$, the solution to (1.1) lies in $C(0, T; H^s)$ for all $T > 0$. There is in general no smoothing action in solving other similar model equations for non-linear dispersive waves in the absence of dissipation (cf. Benjamin and Bona [2]), so the results presented here seem likely to be best possible in terms of the relation of the smoothness of data to the smoothness of the solution. In addition to showing that no smoothness is lost in solving (1.1), it is also demonstrated that the solution depends continuously on the data in the sense that, for all $T > 0$ and $s \geq 2$, the mapping $g \mapsto u$ is a continuous map of H^s into $C(0, T; H^s)$. Thus the initial-value problem (1.1) for the Korteweg-de Vries equation is classically well posed in all the Sobolev spaces H^s for $s \geq 2$.

The proof that the solution to (1.1) lies in $C(0, T; H^s)$ for data in H^s relies on a simple extension of the previously mentioned interpolation theorem of Tartar [11]. The continuous dependence also relies on an interpolation theorem for continuous non-linear operators. These preliminaries concerning abstract interpolation theory are presented in section 2. In section 3, we return to the Korteweg-de Vries equation and apply the ideas of section 2 to obtain the results stated above.

2. An interpolation theorem for non-linear operators.

Recall the K-method (or real method) of interpolation (cf. Butzer and Berens [4]). Let B_0 and B_1 be two Banach spaces such that $B_1 \subset B_0$ with the inclusion

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