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# A GLOBAL REAL ANALYTIC NULLSTELLENSATZ

WILLIAM A. ADKINS AND J. V. LEAHY

## 1. Introduction.

The nullstellensatz problem for an algebra of real or complex valued functions on some space is the problem of determining in some explicit manner all the functions in the algebra which vanish on the zeros of an ideal in the algebra.

The solution of the nullstellensatz problem for the algebra of polynomials over the complex numbers is of course classical. Recently Siu [13] has solved the nullstellensatz problem for the algebra of global holomorphic functions on a Stein space. There have recently been a number of nullstellensatz theorems proved for various algebras of real valued functions. Specifically, there are results for real polynomial rings [7], [9], [11], germs of real analytic functions [12], Nash functions [8], and analytic ideals of differentiable functions [1], [2], [3].

In this paper we investigate the nullstellensatz problem for prime ideals in the algebra of global real analytic functions on a real analytic manifold. The technique employed is a modification of that used by Siu in the complex analytic case. As an application of this general theorem we also prove a better result for the case of principal prime ideals of real analytic functions on manifolds $X$ with $H^1(X, \mathbf{Z}_2) = 0$.

## 2. Preliminaries.

a) Let $A$ be a commutative ring with identity and $\mathfrak{A}$ an ideal of $A$. The ideal $\mathfrak{A}$ is said to be a real ideal if whenever $a_1^2 + \cdots + a_r^2 \in \mathfrak{A}$, $a_i \in A$ it follows that $a_i \in \mathfrak{A}$ for $1 \leq i \leq r$. The ideal $\operatorname{rad}(\mathfrak{A}) = \{a \in A : a^k \in \mathfrak{A}$ for some $k \in \mathbf{N}\}$ will be called the radical of $\mathfrak{A}$. The real radical of $\mathfrak{A}$, denoted $\operatorname{rlrad}(\mathfrak{A})$, is the radical of $\mathfrak{s}(\mathfrak{A})$ where $\mathfrak{s}(\mathfrak{A}) = \{a \in A : a^2 + \sum_{i=1}^r a_i^2 \in \mathfrak{A}$ for some $a_1, \cdots, a_r \in A\}$. It is not difficult to show that $\mathfrak{s}(\mathfrak{A})$ is an ideal, that $\operatorname{rlrad}(\mathfrak{A})$ is the intersection of all real prime ideals of $A$ which contain $\mathfrak{A}$, and that $\mathfrak{A}$ is a real ideal if and only if $\mathfrak{A} = \operatorname{rlrad}(\mathfrak{A})$. If $A$ is a noetherian ring then it can be shown that $\operatorname{rlrad}(\mathfrak{A}) = \mathfrak{o}_1 \cap \cdots \cap \mathfrak{o}_r$ where $\{\mathfrak{o}_1, \cdots, \mathfrak{o}_r\}$ are the minimal real prime ideals of $A$ containing the ideal $\mathfrak{A}$. For information on real ideals and real radicals consult the papers of Dubois–Efroymson [7] and Galbiati–Tognoli [9].

b) Let $\mathfrak{X} = (X, \mathcal{A})$ be a real analytic manifold where $X$ is a second countable Hausdorff topological space and $\mathcal{A}$ the structural sheaf. Then $\mathcal{E}$ will denote the sheaf of germs of $C^\infty$ functions on $X$ determined by $\mathfrak{X}$. If $U$ is an open subset of $X$ let $\mathcal{E}(U) = \Gamma(U, \mathcal{E})$ be the Frechet algebra of real valued $C^\infty$ functions on $U$ with the $C^\infty$ topology. If $E$ is any subset of $\mathcal{E}(U)$ and $Y$ any subset

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