Vol. 43, No. 1 DUKE MATHEMATICAL JOURNAL© March 1976

# LOCALLY SMOOTH CIRCLE ACTIONS ON HOMOTOPY 4-SPHERES

RONALD FINTUSHEL

**1. Introduction.** In this paper we classify up to weak equivalence locally smooth effective actions of $S^1$ on homotopy 4-spheres. The classification is accomplished in terms of some rather minimal orbit data. If any compact connected Lie group other than $S^1$ acts on a homotopy 4-sphere then P. Orlik has shown in [5] that the homotopy 4-sphere is actually $S^4$. In [6] R. W. Richardson has shown that actions on $S^4$ of compact connected Lie groups of dimension at least two are equivalent to linear actions.

Throughout this paper $M^4$ will denote a homotopy 4-sphere with locally smooth $S^1$ action. For any subset $X$ of $M^4$, $X^*$ denotes its image in the orbit space $M^*$. All actions are taken to be effective.

The present investigation may be considered as having originated in [3] where Montgomery and Yang obtained the following information.

(1.1) The fixed point set $F$ is homeomorphic either to $S^2$ or to a pair of points. In the first case $M^*$ is a homotopy 3-cell with boundary $F^*$. In the other case $M^*$ is a homotopy 3-sphere.

(1.2) If $E$ denotes the exceptional orbit set, $F^* \cup E^*$ is polyhedral in $M^*$.

(1.3) There is no simple closed curve $K$ in $E^*$ on which the orbit types are constant.

Actually, these results are proved in the context of differentiable actions of $S^1$ on $S^4$. However the proofs carry over to the present situation.

PROPOSITION 1.4. *There are at most two exceptional orbit types. If there is one exceptional orbit type then $E^* \cup F^*$ is an arc, and $F^*$ is the set of endpoints. If there are two exceptional orbit types then $E^* \cup F^*$ is a simple closed curve separated by $F^*$ into two open arcs on each of which the orbit type is constant.*

*Proof.* If $x \in E$ let the closed 3-disk $S_x$ be a linear slice at $x$. The isotropy group at $x$ is a finite cyclic group $\mathbf{Z}_\alpha$ acting as a group of rotations. $E \cap S_x$ is the axis of rotation, and each point in $E \cap S_x$ has isotropy group $\mathbf{Z}_\alpha$. It follows from (1.3) that $E^*$ consists of a collection of open arcs each of constant orbit type.

If $y \in F$, let the closed 4-disk $S_y$ be a linear slice at $y$. The $S^1$ action on $S_y$ is the cone of the $S^1$ action on $\partial S_y$. If the action on the 3-sphere $\partial S_y$ has fixed points, it has exactly one circle of fixed points and no exceptional orbits [4]. If the action on $\partial S_y$ is fixed point free, it has at most two exceptional orbits. Further, if there are two exceptional orbits on $\partial S_y$ they have orbit types $\mathbf{Z}_{\alpha_1}$ and $\mathbf{Z}_{\alpha_2}$ with $\alpha_1$ and $\alpha_2$ relatively prime [2].

Received May 31, 1975.