Vol. 43, No. 1 DUKE MATHEMATICAL JOURNAL© March 1976

A CHARACTERIZATION OF THE BOREL SUBALGEBRAS OF SEMI-SIMPLE LIE ALGEBRAS AMONG ALL SOLVABLE LIE ALGEBRAS

MICHAEL A. GAUGER

Introduction. This paper is concerned with the theory of finite-dimensional Lie algebras over an algebraically closed field of characteristic zero. For a semi-simple algebra of this type the literature abounds with evidence that a maximal solvable (Borel) subalgebra plays an important role in many aspects of the theory. It is obvious that within the class of all solvable algebras the Borel algebras are quite distinctive objects. Hence it would be interesting to see which of their properties distinguish them from the others.

One must thus look for properties of the Borel algebras which are quite unexpected for an ordinary solvable algebra. A somewhat striking fact is that all their derivations are inner. In fact, the same is true for any parabolic subalgebra (one containing a Borel subalgebra) of a semi-simple Lie algebra. Since there are no totally obvious examples of algebras having only inner derivations (traditionally called complete if in addition the center is trivial) other than the semi-simple ones and their subalgebras mentioned above, one is tempted to conjecture they are the only ones.

In fact, they are not. Just such questions have been extensively and successfully investigated by Leger and Luks. Leger ([6], Prop. 11, 642) has determined the structure of an arbitrary Lie algebra having only inner derivations. Also, a certain construction of solvable Lie algebras is shown to yield a complete algebra ([6], Prop. 10, 641 and [7], Cor. 3.2, 1021, Prop. 3.3, 1021 and Prop. 4.1, 1022) and they seem to have been the first to observe that the Borel algebras are complete ([7], Cor. 4.3, 1022). They have illustrated that all the cohomology groups $H^n(B, B)$ vanish for a large class of solvable algebras B which includes the Borel algebras ([7], §5 and [8], §4). In view of the fact that $H^1(L, L) \simeq \mathrm{Der}(L)/\mathrm{Inder}(L)$ for any Lie algebra L, this is an even stronger result. (For a Lie algebra L, $\mathrm{Der}(L)$ ($\mathrm{Inder}(L)$) refers to the Lie algebra of all derivations (inner derivations).) It is remarked in [8] that $H^n(P, P)$ also vanishes for parabolic subalgebras P by similar techniques. Moreover, a characterization of Borel algebras is included in a more powerful isomorphism theorem of theirs ([8], Th.1, 78).

We proceed here to obtain a characterization of the Borel algebras under weaker hypotheses than Leger–Luks. In section one we give a generator-relation presentation of the parabolic and Borel subalgebras of semi-simple algebras. In

Received June 12, 1975. Revision received September 29, 1975. Partially supported by N.S.F. grant G.P. 43814.