Vol. 43, No. 1 DUKE MATHEMATICAL JOURNAL© March 1976

A PICARD THEOREM FOR HOLOMORPHIC CURVES IN THE PLANE

JAMES A. CARLSON AND MARK GREEN

0. Introduction.

A natural algebro-geometric generalization of the situation dealt with in one variable by Picard's theorem is to consider holomorphic maps $f : \mathbf{C}^k \to \mathbf{P}_n$ which omit a hypersurface D. When $k = n$ and D is smooth or has simple singularities, there is a nice answer—if deg D is $\geq n + 2$, then df is everywhere singular ([1], [2]). For $k < n$, the situation is more complicated at present (see [4] for a discussion), although it is conjectured that for D as above, we should have $f(D)$ forced to lie in an algebraic hypersurface of $\mathbf{P}_n$.

Recently ([3]), it was shown for maps to $\mathbf{P}_2$ that if the dual curve of a curve D has no singularities except ordinary double points and D has genus ≥ 2, then $f(\mathbf{C}^k)$ must be constant. Unfortunately, the dual curve of a generic plane curve has cusps as well as ordinary double points, corresponding respectively to inflectional tangents and double tangents. Our main theorem is that if the dual curve of D has only these singularities and if the number of cusps is less than $2g - 2$ (where g = genus of D), then no non-constant holomorphic map $f : \mathbf{C}^k \to \mathbf{P}_2$ omits D from its image. The preceding hypothesis is equivalent to class $(D) < \frac{1}{2}$ deg (D). (See formulas in section 1).

In section one we recall some standard facts about plane algebraic curves. We will then give two independent proofs of the main theorem—one by differential geometry using negative curvature methods in section two, and one by Nevanlinna theory in section three. Each proof gives a stronger theorem, but in different directions.

The authors wish to thank the Guillemet family for their hospitality while this result was proved.

1. Preliminaries about plane curves.

We recall a few basic facts about plane curves. To $\mathbf{P}_2$ is associated the dual projective space $\mathbf{P}_2^*$ consisting of the set of lines in $\mathbf{P}_2$. In terms of coordinates, the point $[a_0, a_1, a_2]$ is associated to the line $a_0z_0 + a_1z_1 + a_2z_2 = 0$. Given a plane curve $D \subset \mathbf{P}_2$, the dual curve $D^* \subset \mathbf{P}_2^*$ consists of those lines tangent to D. The degree of D^* is called the class of D, here denoted c. The curves D and D^* are birationally equivalent, and the genus of their desingularization is denoted by g.

The singularities of D and D^* are related by Plücker's formulas. They cannot both have only nodes and no worse singularities. However, generically D and

Received October 8, 1975. The authors gratefully acknowledge the help of the N.S.F.