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CONVEX SETS OF OPERATORS ON THE DISK ALGEBRA

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Section 1. Introduction.

Let A denote the disk algebra equipped with the sup-norm. Let P denote the set of bounded linear operators mapping A to A which fix 1 and have norm 1. In [2] Rochberg considered sets of the form

$$K(F, G) = \{T \in P \mid TF = G\},$$

where F and G are inner functions in A, and F is non-constant. $K(F, G)$ is a face of P, i.e., $cU + (1 - c)V \in K(F, G)$ where $U, V \in P$ and $c \in (0, 1)$ implies $U, V \in K(F, G)$. Thus, any extreme point of $K(F, G)$ will be an extreme point of P. Rochberg proved that the real dimension of $K(F, G)$ is always $\leq (m - 1)(m + 1)$ where n and m are, respectively the number of zeros of F and G (counting multiplicity). He was also able to construct extreme points of $K(F, G)$ for certain choices of F and G. The results of [2] do not rule out the possibility of $K(F, G)$ being empty.

In this paper, we extend Rochberg's work by showing that $K(F, G)$ always has an extreme point and by showing that the real dimension of $K(F, G)$ is equal to $(n - 1)(m + 1)$. We also discuss the case where $F = Z^n$ and $G = Z^m$, where Z is the identity function on the unit disk and n and m are integers with $n \geq 1$, $m \geq 0$. In particular, we give a complete description of the set of extreme elements of $K(Z^n, Z)$.

Section 2. $K(F, G)$ has an extreme point.

Let D be the unit disk centered at 0 and let Γ be the boundary of D. We will use $\mathcal{A}$ to denote the sub-algebra of A consisting of functions which can be continued analytically across Γ. Let $f \in \mathcal{A}$ and let γ be a circle centered at 0, having radius > 1 such that f and F are analytic on γ and its interior. Define $T_0 f$ by

$$T_0 f(w) = (2\pi i n)^{-1} \int_\gamma f(\xi) F'(\xi)(G(w) - F(\xi))^{-1}\, d\xi \tag{1}$$

for $w \in \bar{D}$. Note that $T_0 f(w) = n^{-1} \sum_{F(u)=G(w)} f(u)$. $\|T_0 f\| \leq \|f\|$. Hence, the linear operator $f \to T_0 f$ carries $\mathcal{A}$ into A, has norm 1, and satisfies $T_0 F = G$. Since $\mathcal{A}$ is dense in A, it follows that the map $f \to T_0 f$ has a unique extension, denoted by T_0, to all of A. It is clear that $T_0 \in K(F, G)$. We will call T_0 the *center* of $K(F, G)$. We have proved the following:

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