Vol. 42, No. 4 DUKE MATHEMATICAL JOURNAL© December 1975

A NOTE ON RIESZ POTENTIALS

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1. Introduction. The spaces $L^{p,\lambda}$ of C. B. Morrey have been of great value through the years in studying the local behavior of solutions to second order elliptic partial differential equations. Hence several authors have studied the nature of the functions U whose first order distribution derivatives U_x belong to a Morrey space. There is however, one noteworthy gap in this theory: there is no analogue of the Sobolev lemma in the case when U *cannot* be redefined on a set of measure zero to make it a continuous (usually Hölder continuous) function. There are several approximations, the closest due to G. Stampacchia [12]. There it is shown that $U_x \in L^{p,\lambda}$ implies $U \in L_*^{\tilde{p},\lambda}$, i.e. weak—$L^{\tilde{p},\lambda}$, $1/\tilde{p} = 1/p - 1/\lambda$, $0 < \lambda \leq n$: the measure of $\{x \in B \mid U(x) - U_B \mid > t\}$ is dominated by a constant times $t^{-\tilde{p}} |B|^{1-\lambda/n}$, B = ball in $\mathbf{R}^n$ and $|B|$ = its measure. This misses the mark, however, since when $\lambda = n$ we expect the usual Sobolev lemma, namely $U \in L^{\tilde{p},n} = L^{p^*}$, $1/p^* = 1/p - 1/n$, $1 < p < n$, but are presented with only a "weak type" estimate in the sense of Marcinkiewicz. Thus two drawbacks are apparent: first, the need for a new proof that will allow higher order derivatives and second, to determine if indeed the "strong type" estimates are possible when $\lambda < n$ and $p > 1$. These two objections are answered by Theorem 3.2 below by obtaining estimates for Riesz potentials interms of the maximal functions

$$M_\beta f(x) = \sup_{r>0} r^{\beta-n} \int_{|x-y|<r} |f(y)|\, dy \qquad 0 \leq \beta \leq n.$$

These "fractional" maximal functions have already been used by B. Mukenhoupt—R. Wheeden [10] and G. Welland [15] in studying Riesz potentials with regard to weighted norm inequalities of the Sobolev type. The estimates given here show that these maximal functions are indispensible for studying Riesz potentials in L^p, but are *not* a sharp tool for analyzing their pointwise behavior. These estimates appear in section 3.

In section 4, an application of the techniques of section 3 is given in the form of an estimate for a parabolic L^p-capacity. In section 5, a trace theorem for Riesz potentials of functions in a Morrey space is given. The difference between this result and Theorem 3.2 is that at present there is no Marcinkiewicz interpolation theorem for operators on Morrey spaces.

Received March 11, 1975. Revision received August 8, 1975. Research partially supported by National Science Foundation Grant GP-42963.