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# A COUNTEREXAMPLE TO A CONJECTURE OF HOPF

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## 1. Introduction.

In one of a series of classical papers on ergodic theory, Hopf [4] discusses various mixing conditions that a measure-preserving flow $\{T_t\}$ on a Lebesgue measure space $(X, \mu)$ may obey. We list these in increasing order of strength. The flow is *ergodic* (Hopf's "metrically transitive") if whenever a measurable set $A$ is invariant under every transformation $T_t$, then $\mu(A) = 0$ or $\mu(X \backslash A) = 0$; it is *totally ergodic* ("completely transitive") if every $T_t$ with $t \neq 0$ is ergodic; it is *weakly mixing* if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\mu([T_t A] \cap B) - \mu(A)\mu(B)| \, dt = 0$$

for all measurable sets $A$ and $B$; it is *mixing* if

$$\lim_{t \to \infty} \mu([T_t A] \cap B) = \mu(A)\mu(B)$$

for all $A$ and $B$.

In [4] Hopf proves using spectral theory that a flow is totally ergodic if and only if it is weakly mixing, and that these conditions are equivalent to the flow having no eigenfunction with nonzero eigenvalue. Note that the analogous statement for transformations is false; a rotation of the unit circle by an irrational multiple of $2\pi$ has every nonzero power ergodic, but it is not weakly mixing. Hopf also says that he was not able to prove that a totally ergodic flow is mixing, although he has "little doubt about its being true." We construct here a counterexample.

A weakly mixing transformation that is not mixing was first constructed by Kakutani and von Neumann (unpublished). Chacon [2] gave a different and more geometric construction, and using ideas suggested by it showed that the speed of any ergodic flow could be altered to yield a weakly mixing flow. We show here how to make a continuous version of Chacon's construction.

## 2. The construction.

Our flow will be constructed by a continuous analogue of the cutting and stacking constructions of transformations (see [3; Chapter 6]). Let $X_0$ denote the unit square $[0, 1) \times [0, 1)$ equipped with planar Lebesgue measure $\mu$. The flow $\{T_t\}$ is partially defined on $X_0$ by flowing a point at unit speed vertically until it reaches the top; symbolically, $T_t(x, y) = (x, y + t)$ for $-y \leq t < 1 - y$.

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