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BANACH SPACES WHICH ARE ASPLUND SPACES

I. NAMIOKA AND R. R. PHELPS

Dedicated to the memory of Edgar Asplund

A real Banach space E is called an *Asplund space* if every continuous convex function defined on an open convex subset of E is Fréchet differentiable on a dense G_δ subset of its domain. Asplund [1] called these "strong differentiability spaces" and he proved [1, Prop. 5] that the dual E^* of such a space is what we call a (DA)-*space*, that is E^* satisfies the following:

(DA) If C is any weak* compact convex subset of E^*, then C is the weak* closed convex hull of those points in C which are strongly exposed by a functional from E.

We say that $f \in C$ is *strongly exposed by* $x \in E$, $||x|| = 1$, provided

$$M(x, C) \equiv \sup\{g(x) : g \in C\} = f(x)$$

and $||f_n - f|| \to 0$ whenever $f_n \in C$ and $f_n(x) \to f(x)$. Points $f \in C$ with this property will be called *weak* strongly exposed points* of C; they form a (generally proper) subset of the set of extreme points of C.

One of the main theorems (Theorem 6) in the present paper is the converse to Asplund's result: If E^* is a (DA)-space, then E is an Asplund space. This characterization of Asplund spaces in terms of their duals makes it possible to prove a number of permanence properties for this class of spaces, properties which so far have seemed to be intractable. We show, for instance, that subspaces of Asplund spaces are again Asplund spaces and that the c_0 product and the l_p product ($1 < p < \infty$) of any family of Asplund spaces are Asplund spaces. (Asplund [1] showed that this class of spaces is preserved under quotient maps.) It is known that if E^* is separable [1] or if E is reflexive [17], then E is an Asplund space. We extend these results by showing the same conclusion holds if E^* is weakly compactly generated (Corollary 7), a result obtained independently, using a different proof, by Collier [4]. (Collier's proof can also be used to give an alternative proof of Theorem 6.)

Since Fréchet differentiability of a function depends on the topology of E and not upon the particular norm which induces that topology, *the property of being an Asplund space is invariant under isomorphisms.* As shown in [1], if E admits an equivalent norm for which the dual norm is locally uniformly

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