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CHARACTERIZATION OF TEMPERATURES WITH INITIAL DATA IN BMO

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Introduction.

We shall consider real-valued C^∞ functions $u = u(x, t)$ on $\mathbf{R}_+^{n+1} = \mathbf{R}^n \times (0, \infty)$ satisfying the heat equation,

$$Hu \equiv \Delta_x u - u_t = 0 \quad \text{on} \quad \mathbf{R}_+^{n+1},$$

and subject to the identification $u_1 \sim u_2 \Leftrightarrow u_1 - u_2 = \text{constant}$. These solutions will be called *temperatures*, or caloric functions, and we note at once that they have the following properties:

$$\nabla u(x, t) \equiv 0 \Leftrightarrow \nabla_x u(x, t) \equiv 0, \quad \text{where} \quad \nabla = (\nabla_x, D_t), \tag{0.1}$$

$$H(u^2) = 2\,|\nabla_x u|^2. \tag{0.2}$$

We shall denote by $\Gamma(x, t)$, or $\Gamma_t(x)$ with $t > 0$, the elementary solution of the heat operator H: that is, for $t > 0$ and $c_n = (4\pi)^{-n/2}$,

$$\Gamma(x, t) \equiv \Gamma_t(x) = c_n t^{-n/2} \exp\,(-|x|^2/4t)$$

is the Gauss–Weierstrass kernel on $\mathbf{R}^n$. This positive kernel satisfies the estimates:

$$\Gamma(x, t) \le \begin{cases} Ct^{-n/2} \\ C\,|x|^{-n} \end{cases} \tag{0.3}$$

$$|\nabla_x \Gamma(x, t)| \le \begin{cases} Ct^{-(n+1)/2} \\ C\,|x|^{-(n+1)} \end{cases} \tag{0.4}$$

$$|D_t \Gamma(x, t)| \le \begin{cases} Ct^{-(n+2)/2} \\ C\,|x|^{-(n+2)} \end{cases} \tag{0.4'}$$

and so on, for various constants $C > 0$, depending only on n; its Fourier transform in the x variables is given by:

$$[\mathfrak{F}_x \Gamma_t](\xi) = \exp(-t\,|\xi|^2/4). \tag{0.5}$$

With each temperature u on $\mathbf{R}_+^{n+1}$ we may associate the "caloric g-function" $\gamma = \gamma_u$ on $\mathbf{R}^n$, $0 \le \gamma_u(x) \le \infty$, given by

$$[\gamma(x)]^2 = \int_0^\infty |\nabla_x u(x, t)|^2\, dt. \tag{0.6}$$

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