Vol. 42, No. 4 DUKE MATHEMATICAL JOURNAL© December 1975

CRITERIA FOR FINITE GENERATION OF IDEALS OF DIFFERENTIABLE FUNCTIONS

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Introduction.

If J is a closed finitely generated ideal of differentiable functions then it is known that (1) there is a function f in J which satisfies a Lojasiewicz inequality [8]; (2) there are no flat points of J, i.e., there are no points x such that every f in J is flat at x [10]; and (3) the spectrum $V(J)$ of J is the closure of a locally closed submanifold [12]. There has been some effort to try to find which finitely generated ideals of differentiable functions are closed. In dimension one it is known that condition (1) is sufficient [10]. In higher dimensions there is the theorem of Malgrange which states that the ideals which are generated by finitely many analytic functions are closed [8], [12].

In this paper we will present some results related to the question of determining which closed ideals of differentiable functions are finitely generated. As one would expect more conclusive results are obtained when the closed ideal is in fact generated by real analytic functions. An application of these results gives an improved version of the nullstellensatz for analytic ideals of differentiable functions [1].

§1. Preliminaries.

Let $X = (X, \mathcal{E})$ be a C^∞ manifold where X is a second countable hausdorff topological space and $\mathcal{E}$ the structual sheaf. If U is an open subset of X let $\mathcal{E}(U) = \Gamma(U, \mathcal{E})$ be the Frechet algebra of real valued C^∞ functions on U with the C^∞ topology. If $J \hookrightarrow \mathcal{E}(U)$ is an ideal then the Whitney spectral theorem for manifolds [2] shows that the closure of J in $\mathcal{E}(U)$ is given by $\bar{J} = \bigcap_{x \in V(J)} (T_x^U)^{-1}(T_x^U(J))$ where $T_x^U : \mathcal{E}(U) \to \mathfrak{F}_x$ is the Taylor map at x of $\mathcal{E}(U)$ onto the $\mathbf{R}$-algebra of formal power series $\mathfrak{F}_x$ at $x \in U$ and $V(J) = \{x \in U \mid f(x) = 0 \ \forall f \in J\}$.

The following lemma of Tougeron [11] is essential.

Lemma (Tougeron). *If U is an open subset of X and $\{f_i\}_{i \in N}$ a countable family in $\mathcal{E}(U)$ then there exists $\alpha \in \mathcal{E}(X)$ such that*

a) *α is flat on $X - U$ and $\alpha(x) \neq 0 \ \forall x \in U$*

b) *$\alpha f_i \in \mathcal{E}(U)$ extends to a C^∞ function on X which is flat on $X - U \ \forall i \in N$. Any $\alpha \in \mathcal{E}(X)$ satisfying a) and b) above will be called a flat multiplier for the family $\{f_i\}_{i \in N}$.*

Received April 12, 1975.