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OPTIMAL RECONSTRUCTION OF A FUNCTION FROM ITS PROJECTIONS

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§1. Introduction.

Let $f(x, y)$ be square integrable and supported on the unit disk C. The projection $P_f(t, \theta)$ of f is the integral of f along the line $L(t, \theta)$: $x \cos \theta + y \sin \theta = t$. We find an explicit formula for the unique function $g(x, y)$ supported on C and of minimum L_2 norm, which has the same projections as f in each of n equally spaced directions or views, i.e., $P_g(t, \theta_j) = P_f(t, \theta_j)$, for all t and n equally spaced $\theta_j = j\pi/n$, $j = 0, 1, \cdots, n - 1$. We also show that the unique polynomial $P(x, y)$ of degree $n - 1$ which best approximates f in $L^2(C)$ is determined from the above n projections of f, and give a relatively simple explicit formula for P. The exact conditions on n functions $P_j(t)$, $j = 0, \cdots, n-1$, to be the n projections $P_f(t, \theta_j)$ of some $f \in L^2(C)$ are found.

These questions arise naturally in attempting to reconstruct the density $f(x, y, z)$ of an object, in each cross-sectional $x - y$ plane with $z = z_0$ fixed, from measurements of $P_f(t, \theta)$ obtained by passing a thin beam of x-rays along lines $L(t, \theta)$ in the $z = z_0$ plane. In the case treated here the x-ray beam is considered to move discretely in θ and then to translate continuously in t.

In a similarly motivated but different situation, considered by R. B. Marr [M], it is supposed that the projections of f are known over the $N(N - 1)/2$ lines which join each pair of N equally spaced points on the circumference of C. Marr found an explicit formula for the *polynomial* $P^{(M)}(x, y)$ of degree $M \leq N - 2$ whose integrals along the given lines best matches the $N(N - 1)/2$ given projections in the sense of minimizing the sum of squares of the differences. He also studied the case where all projections $P(t, \theta)$ are known and, among other results, found the exact conditions for a function $Q(t, \theta)$ to be the projections $P_f(t, \theta)$ for some $f \in L^2(C)$.

Marr's criterion for optimality has the form of finding the *polynomial* g whose projections match certain finitely many given projections with minimum error ϵ. There is of course no reason to restrict g to be a polynomial. In fact if the degree M of the polynomial is $\geq N - 2$, the error ϵ can be made zero. For a general function g, even for the case considered here where *all* line integrals in each of n views are given, there are many functions g with the exact given projections (assuming the given values are actually the projections of some function). One must give further conditions on g to determine it uniquely. Here we use the criterion of minimizing the L_2 norm of g because (a) this allows

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