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ON TAUT-LEVEL $R\langle x\rangle$

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A domain R is said to be taut-level if height P + depth P = dim R for all primes P. In [13] Ratliff shows that a local domain is taut-level if and only if it satisfies the first chain condition. It is an open question whether the same is true of a semi-local domain (if so many interesting consequences follow, including the Catenary Chain Conjecture. A study of the semi-local case is made in [15]). A recent example of K. Fujita shows that this equivalence fails in a general Noetherian domain. Specifically in [2] he exhibits a Noetherian Hilbert domain R such that with x an indeterminate, $R[x]$ is taut-level but fails to satisfy the first chain condition.

In this paper, our main object is (in essence) to study this question for $R[x]$ when R is a semi-local domain. (In a sense this is the opposite extreme from Fujita's, since his Hilbert domain must contain an abundance of maximal ideals.) The only obstacle is that with R a semi-local domain it is impossible to have $R[x]$ taut-level since there will always be maximal ideals whose height is less than dim $R[x]$ (namely those maximals whose intersection with R is non-maximal). Thus the question applied to $R[x]$ is void.

However let us discard the offending maximals by letting S be the complement in $R[x]$ of the union of those maximals whose intersection with R is maximal and use $R\langle x\rangle$ to denote $R[x]_S$. We then prove that if R is a semi-local domain then $R\langle x\rangle$ is taut-level if and only if it satisfies the first chain condition.

We then take a second look at a famous example of Nagata's.

Preliminaries. R will always be a commutative Noetherian domain with 1, not a field and x will be an indeterminate over R. If P is prime in R and Q is prime in $R[x]$ satisfying $Q \cap R = P$ but $Q \neq PR[x]$, we will call Q an upper to P (or an upper to P in $R[x]$ if clarification seems necessary). We assume familiarity with elementary knowledge of the uppers to P such as is given in [5, Section 1–5]. If T is an integral extension domain of R and if Q is an upper to P in $R[x]$, we will freely use facts concerning the primes of $T[x]$ which lie over Q. Most of them are straightforward and all of them follow easily from [7, Theorem 2].

If $0 \subset P_1 \subset P_2$ are primes and height $P_1 = 1 =$ height (P_2/P_1) we will say that $0 \subset P_1 \subset P_2$ is saturated. We will also say in this case that little height $P_2 = 2$.

It follows from [5, Theorems 24 & 146] that if R is a Noetherian domain and Q is maximal in $R[x]$ then depth $(Q \cap R)$ is either 0 or 1. Let us call Q a type I maximal prime of $R[x]$ if depth $(Q \cap R) = 0$ and a type II maximal prime of $R[x]$ if depth $(Q \cap R) = 1$. Of course by [5, Section 1–3] R is Hilbert if and