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DEGENERATE AND NON-DEGENERATE GROUND STATES FOR SCHRÖDINGER OPERATORS

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1. Introduction.

In the study of quantum mechanical energy operators, particular attention is paid to the question of whether the bottom of the spectrum is an eigenvalue and whether that eigenvalue is simple. The corresponding eigenvector is usually called the gound state. Throughout our discussion we will say that a self-adjoint operator H has no ground state degeneracy when either $E =$ inf spectrum H is not an eigenvalue or E is a simple eigenvalue. Our concern in this note will be Schrödinger operators $H = -\Delta + V$ acting in $L^2(\mathbf{R}^n)$, where V is a suitable multiplication operator.

When $n = 1$ the lack of ground state degeneracy for a large class of V is a classical theorem in the theory of ordinary differential equations. There is an argument in Courant and Hilbert [4] concerning the case of general n. Implicit in their arguments are assumptions (about the regularity of nodes of any eigenvector) that seem difficult to prove for general V. (However, we should remark that a proof along the lines of Courant and Hilbert should be possible, especially if one uses Kato's inequality [14].) The modern treatment of the non-degeneracy problem follows an idea of Glimm and Jaffe [11] in their study of quantum field models. They suggested applying theorems of Perron–Frobenius [10, 17] type to $\exp(-tH)$. The applicability of these ideas to Schrödinger operators was noted by Simon and Höegh-Krohn [23] and Faris (see [2]). It was discussed further in Faris [8], where the following general result appears.

THEOREM 1. *Let* $V = V_1 + V_2$ *where* V_1 *and* V_2 *are real functions on* $\mathbf{R}^n$. *Assume that* $V_1 \geq 0$, $V_1 \in L_{loc}{}^1$, *and that* $V_2 \in L^\infty + L^p$ *where* $p = n/2$ *if* $n \geq 3$, $p > 1$ *if* $n = 2$, *and* $p = 1$ *if* $n = 1$. *Then* $H = -\Delta + V$ *has no ground state degeneracy.*

Remarks. 1. Here and below we define $H = -\Delta + V$ as a sum of quadratic forms. That is, H is the unique self-adjoint operator whose quadratic form domain $\mathcal{Q}(H) \equiv \mathcal{D}(|H|^{\frac{1}{2}})$ is $\mathcal{Q}(-\Delta) \cap \mathcal{Q}(V)$ with

$$\langle (\operatorname{sgn} H)\, |H|^{\frac{1}{2}}\psi, |H|^{\frac{1}{2}}\psi\rangle = \langle \nabla\psi, \nabla\psi\rangle + \langle (\operatorname{sgn} V)\, |V|^{\frac{1}{2}}\psi, |V|^{\frac{1}{2}}\psi\rangle$$

2. The conditions on V_2 are made so that V_2 is a small form perturbation

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