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POSITIVE DEFINITE FUNCTIONALS, FUNCTION-SPACE TRANSFORMS AND ABSTRACT WIENER SPACES

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1. Introduction. Let $\phi(x)$ be a positive definite, continuous functional on a real separable Banach space B, such that $\phi(0) = 1$. In a finite dimensional space such a function can be identified by Bochner's theorem with the Fourier–Stieltjes transform of a positive measure on the space. In particular, Bochner's theorem gives us a simple integral criteria for determining if a continuous function is positive definite, namely a continuous integrable function is positive definite if its Fourier-transform is nonnegative. In this paper we develop analogous criteria for a functional ϕ to be positive definite on a Banach space in terms of the positivity of certain function-space transforms.

Various analogues to Bochner's theorem have been proven for Banach spaces (e.g. [4] and [7]). The best results have been obtained for separable Hilbert spaces by Gross [5] and Sazonov [8]. Because the analysis is much simplified in Hilbert space, we first will derive our results for Hilbert-space and then extend some of these results to Banach space. The extension to Banach space will be accomplished by use of the concept of abstract Wiener spaces, also introduced by Gross [6].

It is well known that there is no function-space analogue to Lebesgue (or Haar) measure, and that therefore a Fourier transform in function-space cannot be readily defined. A Fourier–Wiener transform has been defined however (see [2] and [9]), with properties resembling those of the finite dimensional $\mathfrak{L}^2$ Fourier transform. While we are able to express the criteria for positive definiteness in terms of the positivity of the Fourier–Wiener transform with parameter t (see Corollary 1.2), in order to get the broadest possible results we use a different approach to the Fourier transform.

Let ρ be an operator on a real separable Hilbert space which is linear, symmetric, positive, compact and trace class. It is well known that there is a Gaussian process $\{z \mid z \in H\}$ with mean zero and covariance ρ (i.e. $E_z^{\rho}\{(x, z)^2\} = (x, \rho x)$, where $(\ ,\ )$ denotes the dot product on H, and $E_z^{\rho}\{\ \}$ denotes expectation with respect to $\{z \mid z \in H\}$).

We define for $t \in [0, \infty)$, ϕ continuous and bounded on H

$$T_t\phi(y) = E_z^{t\rho}\{\phi(z) \exp\{i(z, y)\}\}\sqrt{D_\rho(-t)} \tag{1.1}$$

where $D_\rho(-t) = \det(I + t\rho)$. As t tends to infinity, the transform resembles an infinite dimensional Fourier transform, as far as positivity is concerned.

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