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THE CONNECTION BETWEEN THE LAGRANGE AND MARKOFF SPECTRA

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1. Introduction. For any real number θ, define $\mu(\theta)$ by

$$\mu(\theta)^{-1} = \liminf_{q\to\infty} |q(q\theta - p)|$$

where p and q are integers. The set of values taken by $\mu(\theta)$ as θ varies is called the Lagrange spectrum. For any indefinite quadratic form $f(x, y)$ with determinant 1, define $m(f)$ by

$$m(f)^{-1} = \inf |f(x, y)|,$$

where x, y runs through all pairs of integers not both zero. The set of values taken by $m(f)$ as f varies is called the Markoff spectrum. The purpose of this paper is to study the relationship between the two spectra.

It is well known (for instance, see Perron [12]) that the two spectra can also be defined in terms of sequences of positive integers, as follows: Let S denote a doubly infinite sequence $\cdots, a_{-i}, \cdots, a_{-1}, a_0, a_1, \cdots, a_i, \cdots$ of positive integers and define for each integer i

$$\lambda_i(S) = [a_i, a_{i+1}, \cdots] + [0, a_{i-1}, a_{i-2}, \cdots]$$

(here we use the customary notation $[c_0, c_1, c_2, \cdots]$ for the simple continued fraction whose partial quotients are $c_0, c_1, c_2, \cdots$, where c_0 is an integer and the c_i, $i \geq 1$, are positive integers). Further define $L(S) = \limsup \lambda_i(S)$ and $M(S) = \sup \lambda_i(S)$, where the lim sup and sup are both taken over all integers i. As S runs through all possible doubly infinite sequences of positive integers, the set of values taken by $L(S)$ or $M(S)$ is, respectively, the Lagrange or the Markoff spectrum. We shall consider the two spectra from this point of view throughout this paper.

From now on we let **L** and **M** denote, respectively, the Lagrange and Markoff spectra.

It is known and easy to prove that **M** contains **L**. Apparently the first published proofs of this were by Vinogradov, Delone and Fuks [5] and by Vinogradov and Delone [6].

In fact, **M** strictly contains **L**. This was first proved by Freiman [7], who gave an explicit example of a number in **M** but not in **L**.

Thus in order to elucidate the connection between **L** and **M**, it is necessary to describe the conditions under which a number can be in **M** but not in **L**.

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