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INVARIANT SUBSPACES FOR ALGEBRAS GENERATED BY STRONGLY REDUCTIVE OPERATORS

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Let H be a complex separable Hilbert space and T be a bounded linear operator on H. Denote by $\mathcal{A}(T)$ the uniformly closed, inverse-closed algebra generated by $\{I, T\}$ and by $\tilde{T}$ the coset determined by T in the Calkin algebra. The operator T will be called *strongly reductive* (see [6]) if for any $\epsilon > 0$ there exists $\delta > 0$ such that the implication

$$||PTP - TP|| < \delta => ||TP - PT|| < \epsilon$$

holds for all orthogonal projections P.

Recall that an algebra $\mathcal{A}$ of operators acting on H is called intransitive if there exists a proper subspace invariant for all operators belonging to $\mathcal{A}$.

The aim of this note is to prove the following:

THEOREM. *If T is strongly reductive and if $\mathcal{A}(T)$ contains an operator S such that $||S|| \neq ||\tilde{S}||$, then $\mathcal{A}(T)$ is an intransitive algebra.*

Our theorem is similar to a result of Meyer-Nieberg [7] or C. Pearcy and N. Salinas [8], which could be stated as follows: If T is quasitriangular (in the sense of Halmos [5]) and if $\mathcal{A}(T)$ contains a compact operator S, then $\mathcal{A}(T)$ is an intransitive algebra.

As we shall see below, the assumption that T is strongly reductive implies that T is quasitriangular, but on the other hand, we allow S to be non-compact. Also we employ the similar techniques used by Arveson-Feldman [3], Meyer-Nieberg [7], and Pearcy-Salinas [8]; namely, the approximation approach initiated by Aronszajn and Smith [2].

In order to prove our theorem we shall need the following:

PROPOSITION. *Suppose that T is quasitriangular and that $\mathcal{A}(T)$ contains an operator S such that $||S|| \neq ||\tilde{S}||$. Then there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank projections such that:*

1. $||(I - P_n)LP_n|| \to 0$ *for every L in $\mathcal{A}(T)$,*
2. $P_n \xrightarrow{w} A$ (i.e. P_n *weakly tends to* A),

and

3. *A is not a scalar multiple of I.*

Proof. Let E be the spectral measure of $(S^*S)^{1/2}$. Fix some $\rho > 0$ such that $||\tilde{S}|| < \rho < ||S||$. Put $S_\rho = SE([\rho, ||S||])$. It is easy to see that $||S|| \in \sigma_p\{(S^*S)^{1/2}\}$, S_ρ is finite-rank and $S^*S_\rho = S_\rho^*S_\rho$. Fix a positive number

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