Vol. 42, No. 3 DUKE MATHEMATICAL JOURNAL© September 1975

THE CAUCHY PROBLEM WITH POLYNOMIAL GROWTH CONDITIONS FOR PARTIAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

R. B. MELROSE

Suppose $P(D)$ is a linear partial differential operator with constant coefficients in $\mathbf{R}^n$ and $H_N = \{x \in \mathbf{R}^n ; x.N \geq 0\}$, where $N \in \mathbf{R}^n\backslash\{0\}$, is a half-space. In [4], Hörmander has obtained conditions on the zeros of the characteristic polynomial P of $P(D)$ equivalent to the solvability of the distributional Cauchy problem for $P(D)$ with respect to H_N. The operators satisfying these conditions are said to be evolution operators, with respect to H_N. Thus, if $f \in \mathcal{E}'(\mathbf{R}^n)$ has support in a half-space H_N with respect to which $P(D)$ is an evolution operator, there exists at least one distribution $u \in \mathcal{D}'(\mathbf{R}^n)$ with supp $(u) \subset H_N$ such that $P(D)u = f$. In general each such u has exponential growth in some direction parallel to the hyperplane ∂H_N. Here we shall find conditions on the zeros of P equivalent to the existence of a forward fundamental solution with polynomial growth.

DEFINITION. A partial differential operator $P(D)$ with constant coefficients on $\mathbf{R}^n$ is said to be a temperate-evolution operator with respect to the half-space H_N if it has a fundamental solution $E \in \mathcal{D}'(\mathbf{R}^n)$ satisfying

(a) supp $(E) \subset H_N$

(b) there exists a $c \in \mathbf{R}$ such that exp $(-cx.N)E(x) \in \mathcal{S}'(\mathbf{R}^n)$.

It is clear from the results in [2] that such a 'temperate' forward fundamental solution cannot have, in general, the maximal regularity property which is known to hold for some forward fundamental solution (that is one with support in H_N).

If $T \in \mathbf{R}$ put $H_N(T) = \{x \in \mathbf{R}^n ; x.N \geq T\}$ and let $\mathcal{S}'(H_N(T))$ and $\mathcal{D}'(H_N(T))$ denote, respectively, the closed subspaces of $\mathcal{S}'(\mathbf{R}^n)$ and $\mathcal{D}'(\mathbf{R}^n)$ of distributions supported by $H_N(T)$. Any $P(D)$ defines a continuous map of $\mathcal{S}'(H_N(T))$ into itself and so, if $T > 0$, defines a map of $\mathcal{S}'(H_N)/\mathcal{S}'(H_N(T))$ into itself.

THEOREM. *If $P(D)$ is a partial differential operator with constant coefficients on $\mathbf{R}^n$, and $N \in \mathbf{R}^n\backslash\{0\}$ the following three conditions are equivalent.*

(i) *$P(D)$ is a temperate-evolution operator with respect to H_N.*

(ii) *There exists a $T > 0$ and a compact set $K \subset H_N\backslash H_N(T)$ with non-empty interior such that*

$$C_0^\infty(K) \subset P(D)\{\mathcal{S}'(H_N)/\mathcal{S}'(H_N(T))\}.$$

Received April 1, 1975.