DISINTEGRATION OF MEASURES AND THE
VECTOR-VALUED RADON-NIKODYM
THEOREM

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1. Introduction.

Classical vector-valued Radon–Nikodym theorems include [2] [9] [11]. In
the last ten years, several other general vector-valued Radon–Nikodym theorems
have been proved ([10] [12] [13] [7] [5] among many others). Such a theorem
is used here to prove a general disintegration theorem for measures.

The main results (Theorem 3.1 and Corollary 3.3) are as follows. If (X, \( \mathcal{B}(X) \))
is a topological space with its Baire sets, if \( (S, \mathcal{F}) \) is an arbitrary measurable
space, and if \( \mu \) is a probability measure on \( (S \times X, \mathcal{B}(X)) \) whose projection
on \( X \) is tight, then \( \mu \) has a strict disintegration with respect to the projection
on \( S \). If \( (X, \mathcal{B}(X)) \) and \( (S, \mathcal{F}) \) are as above, if \( p : X \to S \) is a measurable function,
and if \( \nu \) is a tight Baire probability measure on \( X \), then \( \nu \) has a disintegration
with respect to \( p \). It is a routine exercise (which we do not do explicitly here)
to extend these results to certain infinite measures. The results are apparently
not special cases of previously known disintegration theorems (see [8, II],
[1, p. 58, Theorem 1], [4, p. 150, Theorem 5], [6]). More importantly, the
proofs given here are quite different (and perhaps simpler) than those referred to.

2. Preliminaries.

Let \( V \) be a locally convex, Hausdorff, topological vector space, and let \( (S, \mathcal{F}, \lambda) \)
be a probability space. A function \( m : \mathcal{F} \to V \) is called a \( V \)-valued measure iff,
for every sequence \( \{E_n\}_{n=1}^\infty \) of disjoint members of \( \mathcal{F} \), we have
\( m(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty m(E_n) \), where the series converges in the topology of the space \( V \).
The measure \( m \) is said to be absolutely continuous with respect to \( \lambda \) iff \( m(E) = 0 \)
for all \( E \in \mathcal{F} \) with \( \lambda(E) = 0 \). The average range of \( m \) is the set \( \{m(E)/\lambda(E) : E \in \mathcal{F}, \lambda(E) > 0\} \).

The Radon–Nikodym theorem we will be using is the following. An element-
ary proof of a more general theorem is given in [5, Theorem 4.9].

2.1 Theorem. Let \( V \) be a locally convex space, and let \( (S, \mathcal{F}, \lambda) \) be a probability
space. Let \( m : \mathcal{F} \to V \) be a measure. Assume (1) \( m \) is absolutely continuous
with respect to \( \lambda \), and (2) \( m \) almost has relatively compact average range, i.e.,
for every \( \epsilon > 0 \), there is \( E_0 \in \mathcal{F} \) such that \( \lambda(E_0) \geq 1 - \epsilon \) and \( \{m(E)/\lambda(E) : E \in \mathcal{F}, E \subseteq E_0, \lambda(E) > 0\} \) is relatively compact. Then there is a function \( \varphi : S \to V \)

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