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DISINTEGRATION OF MEASURES AND THE VECTOR-VALUED RADON-NIKODYM THEOREM

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1. Introduction.

Classical vector-valued Radon–Nikodym theorems include [2] [9] [11]. In the last ten years, several other general vector-valued Radon–Nikodym theorems have been proved ([10] [12] [13] [7] [5] among many others). Such a theorem is used here to prove a general disintegration theorem for measures.

The main results (Theorem 3.1 and Corollary 3.3) are as follows. If $\langle X, \mathcal{B}(X)\rangle$ is a topological space with its Baire sets, if $\langle S, \mathfrak{F}\rangle$ is an arbitrary measurable space, and if μ is a probability measure on $\langle S \times X, \mathfrak{F} \times \mathcal{B}(X)\rangle$ whose projection on X is tight, then μ has a strict disintegration with respect to the projection on S. If $\langle X, \mathcal{B}(X)\rangle$ and $\langle S, \mathfrak{F}\rangle$ are as above, if $p : X \to S$ is a measurable function, and if ν is a tight Baire probability measure on X, then ν has a disintegration with respect to p. It is a routine exercise (which we do not do explicitly here) to extend these results to certain infinite measures. The results are apparently not special cases of previously known disintegration theorems (see [8, §II], [1, p. 58, Theorem 1], [4, p. 150, Theorem 5], [6]). More importantly, the proofs given here are quite different (and perhaps simpler) than those referred to.

2. Preliminaries.

Let V be a locally convex, Hausdorff, topological vector space, and let $\langle S, \mathfrak{F}, \lambda\rangle$ be a probability space. A function $m : \mathfrak{F} \to V$ is called a V-valued *measure* iff, for every sequence $\langle E_n\rangle_{n=1}^{\infty}$ of disjoint members of $\mathfrak{F}$, we have $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$, where the series converges in the topology of the space V. The measure m is said to be *absolutely continuous* with respect to λ iff $m(E) = 0$ for all $E \in \mathfrak{F}$ with $\lambda(E) = 0$. The *average range* of m is the set $\{m(E)/\lambda(E) : E \in \mathfrak{F}, \lambda(E) > 0\}$.

The Radon–Nikodym theorem we will be using is the following. An elementary proof of a more general theorem is given in [5, Theorem 4.9].

2.1 THEOREM. *Let V be a locally convex space, and let $\langle S, \mathfrak{F}, \lambda\rangle$ be a probability space. Let $m : \mathfrak{F} \to V$ be a measure. Assume* (1) *m is absolutely continuous with respect to λ, and* (2) *m almost has relatively compact average range,* i.e., *for every $\epsilon > 0$, there is $E_0 \in \mathfrak{F}$ such that $\lambda(E_0) \geq 1 - \epsilon$ and $\{m(E)/\lambda(E) : E \in \mathfrak{F}, E \subseteq E_0, \lambda(E) > 0\}$ is relatively compact. Then there is a function $\varphi : S \to V$*

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