Vol. 42, No. 3 DUKE MATHEMATICAL JOURNAL© September 1975

IMBEDDINGS OF STIEFEL MANIFOLDS INTO GRASSMANNIANS

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Introduction. A. Weinstein observed in (3) that certain metrics constructed on odd-dimensional spheres by M. Berger and I. Chavel as counterexamples to an extension of a lemma of Klingenberg arise very naturally as the induced metric on distance spheres in complex projective space with the Fubini-Study metric. From Weinstein's observation it is natural to expect that some of the spaces and metrics of [1], namely the metrics on Stiefel manifolds, arise in an analogous fashion from imbeddings of the Stiefel manifolds into Grassmannians, the latter with the canonical Riemannian metric.

In this paper we study a class of equivariant imbeddings of Stiefel manifolds into Grassmannians. The imbedded Stiefel manifold is always contained in an orbit of the isotropy subgroup of the Grassmannian. In the special cases when the image is the full isotropy orbit, the metrics induced on the Stiefel manifold are shown to be among those metrics considered in [1], and it is shown that Einstein metrics occur among them. This briefly summarizes the content of Theorems A, B and C.

A formula is derived for the sectional curvature of these special metrics (Proposition 3), and it is used in the discussion of two sets of examples. It is shown that some of the Einstein metrics on the real Stiefel manifolds have both positive and negative sectional curvatures (Proposition 4). The pinching of the non-constantly curved Einstein metric on S^{4q+3} is computed (Proposition 5) and observed to be positive and approaching zero as q increases.

The author wishes to express thanks to E. Wilson for very helpful conversations about these imbeddings.

§1. Let $\mathbf{F}$ denote the field of real numbers $\mathbf{R}$, or complex numbers $\mathbf{C}$, or quaternions $\mathbf{H}$. Let $\mathbf{F}^{n\times p}$ denote the Euclidean space of all $n \times p$ matrices with entries in $\mathbf{F}$, where n and p are positive integers. Let $\mathbf{F}^{n\times p^*}$ denote the open submanifold consisting of all those matrices of maximal rank. The general linear group $GL(p; \mathbf{F})$ acts on $\mathbf{F}^{n\times p^*}$ on the right by ordinary matrix multiplication. When $n > p$, the Grassmannian $G_{n,p}$ of p-planes in $\mathbf{F}^n$ (i.e. p-dimensional $\mathbf{F}$-subspaces) is the orbit space $\mathbf{F}^{n\times p^*}/GL(p; \mathbf{F})$. Denote the natural projection $\mathbf{F}^{n\times p^*} \to G_{n,p}$ by π.

Let h denote the standard hermitian form on $\mathbf{F}^n$ given by $h(\xi, \eta) = \xi^*\eta$, $\xi, \eta \in \mathbf{F}^n$, where ξ^* denotes the conjugate transpose of the column vector ξ.

Received February 5, 1975. This work partially supported by N.S.F. contract number GP 34710.